

THE MILNOR TRIPLE LINKING NUMBER OF STRING LINKS BY CUT-AND-PASTE TOPOLOGY

ROBIN KOYTCHIEFF

ABSTRACT. In [5] Bott and Taubes constructed knot invariants by integrating differential forms along the fiber of a bundle over the space of knots, generalizing the Gauss linking integral. Their techniques were later used to construct real cohomology classes in spaces of knots and links in higher-dimensional Euclidean spaces [7]. In previous work, we constructed cohomology classes in knot spaces with *arbitrary* coefficients by integrating via a Pontrjagin–Thom construction [13]. We carry out a similar construction over the space of string links, but with a refinement in which configuration spaces are glued together according to the combinatorics of weight systems. This gluing is somewhat similar to work of Kuperberg and Thurston [15]. We use a formula of Mellor [18] for weight systems of Milnor invariants, and we thus recover the Milnor triple linking number for string links, which is in some sense the simplest interesting example of a class obtained by this gluing refinement of our previous methods. Along the way, we find a description of this triple linking number as a “degree” of a map from the 6-sphere to a quotient of the product of three 2-spheres.

1. INTRODUCTION

Let \mathcal{L}_k be the space of k -component string links, i.e., the space of embeddings $f : \coprod_k \mathbb{R} \hookrightarrow \mathbb{R}^3$ which agree with a fixed linear embedding outside of $\coprod_k [-1, 1] \subset \coprod_k \mathbb{R}$. This paper concerns the Milnor triple linking number for string links [19, 20], which is a function from isotopy classes of string links to the integers. Since $H_0(\mathcal{L}_k)$ is generated by isotopy classes of string links, we can view this link invariant as a degree-0 class in the cohomology of \mathcal{L}_3 .

We build on our previous work [13]. This work was inspired by the configuration space integrals of Bott and Taubes [5] and subsequent work based on their methods (especially [7]), which produced real cohomology classes in spaces of knots. In our work, we replaced integration of differential forms by a Pontrjagin–Thom construction. This produced cohomology classes with *arbitrary* coefficients. It is not too difficult to generalize configuration space integrals or the homotopy-theoretic construction of our previous work to spaces of links and string links. Perhaps less obvious is a refinement of this construction in which configuration spaces are glued along their boundaries. Here we show that via this refinement, we recover the Milnor triple linking number for string links. This invariant is already known to be integer-valued, but generalizations of this work should have more novel implications.

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We conjecture that this gluing refinement of our “homotopy-theoretic Bott–Taubes integrals” produces integral multiples of all the Bott–Taubes/Vassiliev-type cohomology classes of Cattaneo, Cotta-Ramusino and Longoni [7], showing that these classes are rational. Modulo an anomaly term, they should also produce all finite-type (a.k.a. Vassiliev) invariants of links with rational coefficients. These items are currently work in progress. In this paper, we will focus only on the specific but interesting example of the Milnor triple linking number, rather than all finite-type invariants or all the cohomology classes of Cattaneo et al.

As our title suggests, this gluing is inspired by and similar to a construction of Kuperberg and Thurston [15, 17]. This gluing idea was also present in the work of Bott and Taubes (in equation (1.18) of [5]), as once pointed out to the author by N. Habegger. The difference between our construction and that of Kuperberg and Thurston is that we do a gluing which is specific to the weight system for the invariant. Thus, in our work in progress, we generalize this to arbitrary cohomology classes by constructing one glued-up space for each weight system (or graph cocycle), rather than a glued-up space that accounts for all weight systems. This slightly different approach is necessary to produce bundles whose fibers are nice enough to admit neat embeddings and Pontrjagin–Thom constructions.

1.1. The Gauss linking integral. Before stating our main results, we will describe the analogue of our constructions in the simpler case of the Gauss linking number. We will first discuss in detail this pairwise linking number in terms of a Pontrjagin–Thom construction and the space of links in the case of *closed* links, where the linking number is completely straightforward. In order to simplify our gluing constructions and thus describe the triple linking number for string links as the degree of a map, we need a definition of string links which is slightly different from the usual one. As a result, even though the triple linking number for string links is a well defined integer, we encounter some indeterminacy. This indeterminacy occurs even in the pairwise linking number. We will ultimately relate our invariant to the usual triple linking number of string links (without any indeterminacy), but we explain the indeterminacy in the simple case of the pairwise linking number to clarify the issue.

For any space X , denote the configuration space of q points in X by

$$C_q(X) := \{(x_1, \dots, x_q) \in X^q \mid x_i \neq x_j \forall i \neq j\}.$$

An inclusion $X \xrightarrow{f} Y$ induces an inclusion of configuration spaces $C_q(X) \xrightarrow{f} C_q(Y)$.

The linking number of $L : S^1 \sqcup S^1 \rightarrow \mathbb{R}^3$ is then given by the degree of the composition

$$(1) \quad S^1 \times S^1 \hookrightarrow C_2(S^1 \sqcup S^1) \xrightarrow{L} C_2(\mathbb{R}^3) \xrightarrow{\varphi_{12}} S^2$$

$$(x_1, x_2) \longmapsto \frac{x_1 - x_2}{|x_1 - x_2|}$$

where $S^1 \times S^1$ is the subspace of configurations where the two points are on distinct circles, and where the rightmost map happens to be a homotopy equivalence. One way of realizing this degree is to pull back a volume form on S^2 to a 2-form θ and integrate θ over $S^1 \times S^1$.

Another way of realizing this integer would be to start by embedding $S^1 \times S^1$ into some Euclidean space \mathbb{R}^N . The Pontrjagin–Thom collapse map then gives a map from S^N to the Thom space $(S^1 \times S^1)^\nu$ of the normal bundle ν of the embedding:

$$S^N \xrightarrow{\tau} (S^1 \times S^1)^\nu$$

The normal bundle ν has fiber dimension $N - 2$, so using the Thom isomorphism we have in integral cohomology

$$H^2(S^1 \times S^1) \xrightarrow{\text{Th} \cong} H^N((S^1 \times S^1)^\nu) \xrightarrow{\tau^*} H^N(S^N) \cong \mathbb{Z}.$$

(The normal bundle over $S^1 \times S^1$ happens to be trivial, so $(S^1 \times S^1)^\nu$ is just $\Sigma^{N-2}(S^1 \times S^1)$, and the Thom isomorphism reduces to the suspension isomorphism.) If we again let θ denote the pullback to $S^1 \times S^1$ of a volume form on S^2 via (1), the image of the cohomology class $[\theta]$ under the above composition is the linking number.

To describe this as a function on the space of links, let $\mathring{\mathcal{L}}_k$ denote the space $\text{Emb}(\coprod_k S^1, \mathbb{R}^3)$ of k -component closed links (as opposed to string links). We can rewrite (1) as

$$(2) \quad \mathring{\mathcal{L}}_2 \times S^1 \times S^1 \hookrightarrow \mathring{\mathcal{L}}_2 \times C_2(S^1 \sqcup S^1) \longrightarrow C_2(\mathbb{R}^3) \longrightarrow S^2.$$

Then we can pull back a volume form on S^2 (or generator of $H^2(S^2)$) to a 2-form θ on $\mathring{\mathcal{L}}_2 \times S^1 \times S^1$ (or a cohomology class $[\theta]$). The latter space is a trivial $(S^1 \times S^1)$ -bundle over $\mathring{\mathcal{L}}_2$, and integrating θ along the fiber gives a function on $\mathring{\mathcal{L}}_2$, which is the linking number.

This integration along the fiber can also be described via a Pontrjagin–Thom construction as follows. We embed the bundle into a trivial \mathbb{R}^N -bundle

$$(3) \quad \mathring{\mathcal{L}}_2 \times S^1 \times S^1 \hookrightarrow \mathring{\mathcal{L}}_2 \times \mathbb{R}^N$$

and collapse a complement of the tubular neighborhood of the embedding to get a map

$$(4) \quad \Sigma^N \mathring{\mathcal{L}}_2 \rightarrow (\mathring{\mathcal{L}}_2 \times S^1 \times S^1)^\nu$$

where the right-hand side is the Thom space of the normal bundle ν of the embedding. Using the Thom isomorphism and suspension isomorphism, we have in cohomology

$$H^2(\mathring{\mathcal{L}}_2 \times S^1 \times S^1) \xrightarrow{\text{Th} \cong} H^N((\mathring{\mathcal{L}}_2 \times S^1 \times S^1)^\nu) \longrightarrow H^N(\Sigma^N \mathring{\mathcal{L}}_2) \xrightarrow{\text{susp} \cong} H^0(\mathcal{L}_2)$$

The image of $[\theta]$ under this composite is the linking number. Those familiar with spectra in stable homotopy theory will realize that the maps (4) for all N can be more concisely rewritten as a map from a suspension spectrum to a Thom spectrum. This was the perspective taken in our previous work [13].

1.2. Gauss linking integral for string links. In this paper we consider string links because the Milnor triple linking number is a well defined integer only for string links. (For closed links, it is only defined modulo the gcd of the pairwise linking numbers.)

Definition 1.1. Fix a positive real number R . We define a *string link with k components* as a smooth embedding $L : \coprod_k \mathbb{R} \hookrightarrow \mathbb{R}^3$ such that

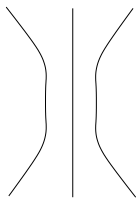
- L takes $\coprod_k [-R, R]$ into $B_R \subset \mathbb{R}^3$, a ball of radius R around the origin
- outside of $\coprod_k [-R, R]$, L agrees with $2k$ fixed linear maps $f_i^- : (-\infty, -R] \rightarrow \mathbb{R}^3$, $f_i^+ : [R, \infty) \rightarrow \mathbb{R}^3$, $i = 1, \dots, k$.

We also require that the direction vectors of any two of the $2k$ fixed linear maps are distinct vectors in S^2 .

Definitions 1.2. We need distinct directions to infinity for defining configuration space integrals later on. In our case of interest $k = 3$, we further require (for simplifications in our construction) that (1) the two ends of each strand agree with the same line through the origin outside the compact set and (2) these three directions to infinity are linearly independent. Call string links (with ≤ 3 components) in such a setting *linearly independent string links*.

More commonly, string links are defined as embeddings agreeing with k fixed parallel, affine-linear maps (e.g., as in Habegger–Lin [10]). The above definition does not include such a definition, but by, say, choosing the $2k$ linear maps to lie in a plane, we can work in a setting, such as the one shown below, where one of our string links corresponds canonically to a string link in the usual parallel definition. (Just “straighten” the parts of the strands outside the fixed compact set.) Call any such setting the setting of *parallel string links*.

FIGURE 1. The unlink in a setting of parallel string links allowed under Definition 1.1.



Suppose we have two settings of string links allowed under Definition 1.1, with constants $R < R'$ and hence $B_R \subset B_{R'}$. We can perform an isotopy relative to ∂B_R from the first fixed linear embedding $\coprod_k ((-\infty, R] \sqcup [R, \infty)) \hookrightarrow \mathbb{R}^3 \setminus B_R$ to an embedding which

on $\coprod_k ((-\infty, R] \sqcup [R, \infty))$ agrees with the second fixed linear embedding. Call such an isotopy a *straightening*. \square

Fix any setting of string links allowed under our definition. A straightening gives a bijection between isotopy classes of these string links and isotopy classes of parallel string links. But the straightening, and hence the bijection, need not be canonical. In particular, there may not be a canonical unlink. There is no canonical unlink for linearly independent string links. As we will shortly see, this causes some indeterminacy which we will fully address in section 3. We could relax “linear” to “affine-linear”, and certain choices of affine-linear embedding would give a canonical unlink; however, the choice of affine-linear embedding with same direction vectors as a fixed linear embedding is equivalent to a choice of unlink.

Now to treat the simplest example in the actual setting of our paper, consider the linking number of a two-component string link. Note that for closed links or parallel string links, this invariant is determined, up to sign, by requiring it to be zero on the unlink and requiring that it takes all integer values. However, our linearly independent string links have no canonical unlink, so there will be choices of pairwise linking number that will differ by constants.

We start by replacing (1) by

$$(5) \quad \mathbb{R} \times \mathbb{R} \hookrightarrow C_2(\mathbb{R} \sqcup \mathbb{R}) \longrightarrow C_2(\mathbb{R}^3) \xrightarrow{\varphi_{12}} S^2.$$

If we compactified $\mathbb{R} \times \mathbb{R}$ to a square disk, the map to S^2 would be ill defined at a corner, taking two different constant values on the intersecting edges. However, we can keep track of the relative rates of approach to infinity of the two points. Such a compactification yields an octagonal disk \bigcirc ; it is just the blowup of the square at the four corners. Thus we have

$$(6) \quad \bigcirc \hookrightarrow C_2[\mathbb{R}^3] \xrightarrow{\varphi_{12}} S^2$$

where $C_2[\mathbb{R}^3]$ is the Axelrod–Singer compactification of $C_2(\mathbb{R}^3)$. (The reader unfamiliar with this compactification may ignore this intermediate space for now and wait until it is reviewed in section 2).

The map $\varphi_{12} : \bigcirc \rightarrow S^2$ takes the boundary of \bigcirc into a great circle C on S^2 , by our definition of string links. Namely, C is the intersection of the sphere with the plane spanned by the directions of the two strands to infinity. We can find a 2-form α on S^2 which vanishes on C but is cohomologous to a unit volume form. We can then pull it back to a 2-form $\theta = \varphi_{12}^* \alpha$ on \bigcirc and integrate θ over \bigcirc .¹ More topologically, we have maps

$$S^2 \cong \bigcirc / \partial \bigcirc \xrightarrow{\varphi_{12}} S^2 / C \cong S^2 \vee S^2 \overset{\text{quotient}}{\longleftarrow} S^2$$

¹Since θ is 0 on the boundary of \bigcirc , one can show (using Stokes’ Theorem for integration along the fiber of the bundle $\mathcal{L}_2 \times \bigcirc \rightarrow \mathcal{L}_2$) that this integral is an isotopy invariant. Cf. equation (14) in section 2.3.

where $[\alpha] \in H^2(S^2/C; \mathbb{Z})$ is a class that maps to a generator of $H^2(S^2; \mathbb{Z})$. The pairing of $[\theta]$ with the fundamental class $[\bigcirc, \partial\bigcirc]$ gives one possible choice ℓ_α for the pairwise linking number.

More concretely, suppose the 2-component link agrees with the x - and y -axes towards infinity, so that C is the equator in the xy -plane. Up to isotopy, there are two equally suggestive candidates L_1, L_2 for the unlink. Fixing a generator $[\omega] \in H^2(S^2; \mathbb{Z})$, there are two integral classes $[\alpha] \in H^2(S^2/C; \mathbb{Z})$ which map to $[\omega]$ (corresponding to the two factors in $S^2 \vee S^2$). One of these gives a linking number which takes ± 1 on L_1 and 0 on L_2 , while the other gives one that takes 0 on L_1 and ∓ 1 on L_2 . The signs depend on the choice of orientation for \bigcirc , but in each case, the difference in the values of the invariant on the two links is the same (± 1). In general, any two choices for the linking number will differ on any fixed link by a constant; Theorem 2.1 guarantees this.² Now for either choice of α , we can quotient by the complement of the support of α :

$$S^2/C \cong S^2 \vee S^2 \xrightarrow{q(\alpha)} S^2$$

Then one can check that the degree of the composition $q(\alpha) \circ \varphi_{12}$ is our linking number ℓ_α .

Moving towards a Pontrjagin–Thom construction with whole space of string links, we parametrize the map (6) as in (2) and pull back a class $[\alpha]$ as above to a class $[\theta]$ on $\mathcal{L}_2 \times \bigcirc$. We can embed

$$\mathcal{L}_2 \times \bigcirc \hookrightarrow \mathcal{L}_2 \times \mathbb{R}^M \times [0, \infty)^N$$

in a way that preserves the corner structure of the octagonal disk \bigcirc . (Alternatively, we could smooth the corners of this disk, since subspaces of codimension > 1 do not affect integration; however, we do not pursue this approach in this paper.) This embedding has a tubular neighborhood diffeomorphic to the normal bundle ν . Collapsing not only the complement of the tubular neighborhood, but also the boundary of $\mathbb{R}^M \times [0, \infty)^N$, gives a collapse map to a quotient of Thom spaces:

$$\Sigma^{M+N} \mathcal{L}_2 \longrightarrow (\mathcal{L}_2 \times \bigcirc)^\nu / (\mathcal{L}_2 \times \partial\bigcirc)^\nu$$

The induced map in cohomology, together with the relative Thom isomorphism and suspension isomorphism, takes the class $[\theta]$ to a pairwise linking number.

1.3. Main Results. One main motivation for this paper was the question of exactly which knot and link invariants can be produced via the Pontrjagin–Thom construction of our previous work [13], or a modification of it. In this paper we begin to answer that question.

²More accurately, the extension of Theorem 2.1 to linearly independent string links guarantees this. (See section 3.1.) We prove this extension only for the triple linking number, but the proof for the pairwise linking number is easier—the previous footnote sketches the proof of part of it.

We glue four different configuration space bundles over the space \mathcal{L}_3 of 3-component string links to obtain one bundle $F_g \rightarrow E_g \rightarrow \mathcal{L}_3$ whose fiber is the gluing of the four configuration space fibers. Along the way to proving a statement about our Pontrjagin–Thom construction, we are led to the following theorem.

Theorem 1. *The Milnor triple linking number for string links can be expressed as the pairing of a 6-dimensional cohomology class $[\beta]$ with the fundamental class of S^6 . This class $[\beta]$ is defined via maps*

$$S^6 \xrightarrow{\Phi} S^2 \times S^2 \times S^2 / \mathcal{D} \longleftarrow S^2 \times S^2 \times S^2$$

where \mathcal{D} is a union of positive-codimension subsets of $S^2 \times S^2 \times S^2$, where the right-hand map is the quotient, and where the left-hand map is induced by the link. Specifically, $[\beta]$ is the pullback via the left-hand map of a certain class $[\alpha] \in H^6(S^2 \times S^2 \times S^2 / \mathcal{D}; \mathbb{Z})$ which maps to a generator of $H^6(S^2 \times S^2 \times S^2; \mathbb{Z})$ via the right-hand map.

Note that since $[\alpha]$ maps to a generator of the top *integral* cohomology of $S^2 \times S^2 \times S^2$, $[\alpha]$ must be supported in a connected component of $S^2 \times S^2 \times S^2 \setminus \mathcal{D}$. Take the quotient $S^2 \times S^2 \times S^2 \xrightarrow{q(\alpha)} S(\alpha)$ by the complement of this component. The resulting quotient space $S(\alpha)$ then has a fundamental class. So we can express the triple linking number as the *degree* of $q(\alpha) \circ \Phi$.

Returning to our Pontrjagin–Thom construction, we show (in Lemma 7.1) that we can embed the bundle E_g into a trivial fiber bundle

$$(7) \quad \begin{array}{ccc} E_g & \hookrightarrow & B_{M,N}(R) \times \mathcal{L}_3 \\ & \searrow & \swarrow \\ & \mathcal{L}_3 & \end{array}$$

in a way that preserves the corner structure. The fiber of this trivial bundle, called $B_{M,N}(R)$, is a codimension-0 subset of a ball of radius R in \mathbb{R}^{M+N} , where the number N is related to the codimension of the corners in E_g . The space $B_{M,N}(R)$ is homeomorphic to a ball, but it is not quite a manifold with corners. Nonetheless, this embedding has a tubular neighborhood $\eta(E_g)$ diffeomorphic to the normal bundle, which is enough for a Pontrjagin–Thom map

$$\tau : (B_{M,N}(R) \times \mathcal{L}_3, \partial(B_{M,N}(R)) \times \mathcal{L}_3) \rightarrow (B_{M,N}(R) \times \mathcal{L}_3, (B_{M,N}(R) \times \mathcal{L}_3 - \eta(E_g)) \cup \partial E_g).$$

Below, we rewrite this as a map of quotients rather than pairs. We let $E_g^{\nu_{M,N}}$ denote the Thom space of the normal bundle $\nu_{M,N} \rightarrow E_g$ to the embedding (7).

Theorem 2. *Let $[\beta]$ be a cohomology class as in Theorem 1 above. The Pontrjagin–Thom collapse map*

$$\tau : \Sigma^{M+N} \mathcal{L}_3 \rightarrow E_g^{\nu_{M,N}} / \partial E_g^{\nu_{M,N}}$$

induces a map in cohomology under which the image of $[\beta]$ in $H^0(\mathcal{L}_3)$ is the Milnor triple linking number for string links .

As mentioned earlier, those familiar with stable homotopy theory will see that this collapse map can be written for all sufficiently large M as a map from a suspension spectrum to a Thom spectrum.

1.4. Organization of the paper. The paper is organized as follows. Section 2 contains a thorough review of finite-type invariants and configuration space integrals, and a brief description of the Milnor triple linking number. This section constitutes a large portion of material, most of which experts can safely skip. However, we discuss in some detail how to define configuration space integrals for string links, a point not previously addressed.

Section 3 describes the triple linking number as a sum of configuration space integrals, using Mellor’s formula for weight systems of Milnor invariants as well as results on configuration space integrals for homotopy string link invariants. We essentially go through the proof of the main theorem of Bott–Taubes integrals for the triple linking number. This is partly because examining the configuration space integrals combine to make an invariant is needed to determine the gluing construction. Another reason for reviewing this proof is that it is not so obvious that all of it extends to our setting of linearly independent string links.

Section 4 starts with some general facts about manifolds with faces, which are a nice kind of manifolds with corners. We then glue the appropriate bundles over the link space \mathcal{L}_3 , and show that the glued space is a manifold with faces.

Section 5 proves Theorem 1, expressing the triple linking number as a degree.

Section 6 shows that integration along the fiber coincides with a certain Pontrjagin–Thom construction. We have left this arguably foundational material to one of the last sections because it is not needed for Theorem 1; in addition, some of the ideas of section 4 are useful for making sense of Pontrjagin–Thom constructions for manifolds with corners.

Finally, section 7 proves Theorem 2, expressing the triple linking number via a Pontrjagin–Thom construction.

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2. BACKGROUND MATERIAL

This section is mainly a review of background material, including finite-type invariants of (string) links, compactifications of configuration spaces, configuration space integrals, and Milnor invariants of string links. The only new material in this section is a clarification of how to define configuration space integrals for string links (section 2.3.2).

2.1. Finite-type invariants. Vassiliev invariants, also known as finite-type invariants, were first defined for knots, but their definition easily extends to links. We will consider only oriented links.³ The ideas below make sense for both closed links and string links, but from here on, we will write “link” to mean “(oriented) string link”.

The simplest definition is via a skein relation, even though this was not Vassiliev’s original definition. Let V be *any* link invariant with values in a field \mathbb{F} (or even a ring R). Thus $V \in H^0(\mathcal{L}_k; \mathbb{F})$. Then V can be extended to singular links with finitely many double-points (transverse self-intersections) via the Vassiliev skein relation

$$V \left(\begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} \right) = V \left(\begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \right) - V \left(\begin{array}{c} \nwarrow \quad \nwarrow \\ \searrow \quad \searrow \end{array} \right)$$

where the arrows denote the orientation of the link components. If V vanishes on links with more than m double points, we say V is a *finite-type invariant of type m* . So if we let \mathcal{V}_m^k be the \mathbb{F} -vector space (or R -module) of type m invariants, then there is a filtration of finite-type invariants

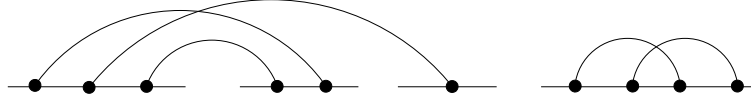
$$\mathbb{F} = \mathcal{V}_0^k \subset \mathcal{V}_1^k \subset \dots \subset \mathcal{V}_m^k \subset \mathcal{V}_{m+1}^k \subset \dots$$

The reader can verify from the skein relation that the pairwise linking number of any 2 components of a k -component link is a type-1 invariant.

The value of a type- m invariant on a singular link with m double points remains unchanged under crossing changes, so it is completely determined by the combinatorial data of the placement of the singularities in the domain. This motivates the definition of a *chord diagram*. A chord diagram on k strands of degree m consists of k labeled, oriented open intervals and m pairs of distinct points on these strands, with each pair joined by an edge, called a *chord*. In our pictures, the strands are always oriented from left to right.

The k strands are thought of as the domain of a singular link and each pair of points is the preimage of a double point. One considers these diagrams up to diffeomorphisms preserving the orientation and labeling of the k strands. Let \mathcal{CD}_m^k be the vector space (or

³Note for example that the pairwise linking number is an invariant of oriented links.

FIGURE 2. A chord diagram in \mathcal{CD}_5^4 .

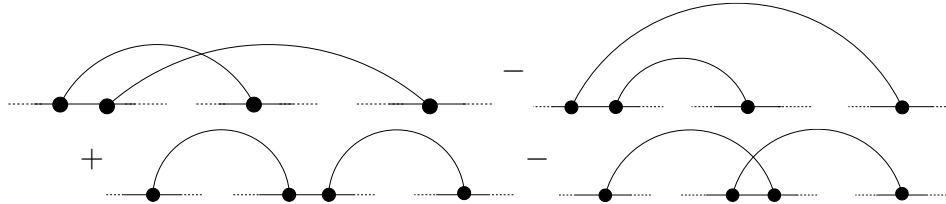
free R -module) of formal linear combinations of equivalence classes of these diagrams. Given a type- m invariant V of k -component links, we get a functional $w(V)$ on chord diagrams, that is, we can define a (linear) map

$$w : \mathcal{V}_m^k \longrightarrow (\mathcal{CD}_m^k)^*$$

as follows. For $D \in \mathcal{CD}_m^k$, put $w(V)(D) := V(L_D)$ where L_D is a k -component singular link with m double points realizing the singularities prescribed by the chords of D . The link L_D is not well defined up to isotopy, but by the remark in the previous paragraph, $V(L_D)$ is well defined. Obviously w sends type $m - 1$ invariants to 0, so w can be regarded as a map

$$(8) \quad w : \mathcal{V}_m^k / \mathcal{V}_{m-1}^k \longrightarrow (\mathcal{CD}_m^k)^*$$

which is easily verified to be injective. Furthermore, any element W in the image of w satisfies the *one-term relation* (1T) and the *four-term relation* (4T). The relations 1T are that W vanishes on any diagram containing a chord joining adjacent points on the same strand. The relations 4T are that W vanishes on all sums of diagrams of the form below, where the four diagrams are the same outside the picture:



The pictured strands have the same labels in all four diagrams, but there are no other restrictions on these labels. E.g., they may or may not be distinct. However, the orientations must be the ones shown (i.e., left to right).

The 1T and 4T relations follow from dual relations that an invariant V must satisfy on singular links, which in turn can be derived from the skein relation. See Bar-Natan's paper [3] for a more detailed explanation.

Call an element in $(\mathcal{CD}_m^k)^*$ satisfying these relations a *weight system*, and let \mathcal{W}_m^k be the subspace of $(\mathcal{CD}_m^k)^*$ of all weight systems. We can alternatively describe \mathcal{W}_m^k as the dual to the quotient of $\mathcal{CD}_m^k / (1T, 4T)$ by the analogues of the relations (1T) and (4T) on the diagrams themselves.

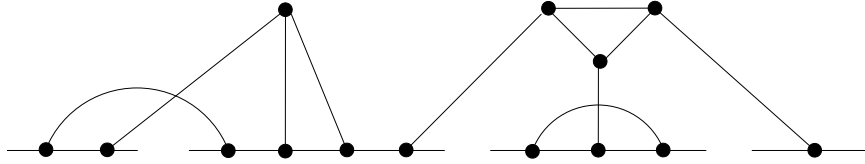
The main theorem in the theory of finite type invariants is that the linear injection w in (8) is in fact an isomorphism of \mathbb{R} -vector spaces:

$$(9) \quad \mathcal{V}_m^k / \mathcal{V}_{m-1}^k \xrightarrow[w]{} \mathcal{W}_m^k$$

One way of constructing the inverse map $\mathcal{W}_m^k \rightarrow \mathcal{V}_m^k / \mathcal{V}_{m-1}^k$ is by the Kontsevich integral.

2.2. Finite-type invariants and trivalent diagrams. Another way of constructing finite-type knot and link invariants from combinatorial data of weight systems is via the configuration space integrals of Bott and Taubes. This approach requires enlarging the vector space of chord diagrams \mathcal{CD}_m^k to the vector space of trivalent diagrams \mathcal{TD}_m^k . The most complete exposition of this diagram space (as well as generalizations and refinements of it) is given in [14]. We review the definition here for the reader's convenience.

FIGURE 3. A trivalent diagram in \mathcal{TD}_7^4 . The horizontal lines are the strands. The Lie orientation is determined by this drawing of the diagram in the plane.



A *trivalent diagram* consists of k oriented open intervals together with $2m$ vertices (not necessarily on the k intervals) and edges between the vertices. The graph resulting from forgetting the intervals must be simple, i.e., self-loops and multi-edges are not allowed. Furthermore, every vertex must be connected by a path to some interval. The vertices on the intervals are univalent and the remaining vertices are trivalent. We call the former *interval vertices* and the latter *free vertices*. If one imagines that the segments of the intervals are edges, then every vertex is trivalent. However, we do not consider interval segments as edges. We call an edge between two interval vertices a *chord*. Finally, these diagrams are equipped with a *Lie orientation*. That means that at every vertex we have a cyclic order of the three emanating edge-ends (say, determined by a planar embedding of these edge-ends), and in \mathcal{TD}_m^k , the effect of reversing the cyclic order at one vertex is to multiply the diagram by -1 . We put a canonical cyclic order at each interval vertex by drawing all edges *above* the strands, so there is a well defined inclusion $\mathcal{CD}_m^k \subset \mathcal{TD}_m^k$. This Lie orientation is equivalent to one determined by an ordering of the vertices and an orientation on every edge, where changing the vertex-ordering by an odd permutation multiplies the diagram by -1 , as does reversing the orientation on one edge. (See Appendix B of [23] or section 3.1 of [15].)

If we impose the STU relation below, then every trivalent diagram can be expressed as a sum of chord diagrams. (Note that the cyclic order at the free vertex, determined by the

planar embedding, matters.)

$$(10) \quad \begin{array}{c} \text{---} \diagup \text{---} \\ \bullet \\ \text{---} \text{---} \text{---} \text{---} \\ \bullet \end{array} = \begin{array}{c} \text{---} \diagup \text{---} \\ \bullet \end{array} - \begin{array}{c} \text{---} \diagdown \text{---} \\ \bullet \end{array}$$

So \mathcal{TD}_m^k/STU is isomorphic to a quotient of \mathcal{CD}_m^k . Bar-Natan showed that $\mathcal{TD}_m^k/STU \cong \mathcal{CD}_m^k/4T$ [3], so $\mathcal{TD}_m^k/(STU, 1T) \cong \mathcal{CD}_m^k/(4T, 1T)$. Thus $\mathcal{W}_m^k \cong (\mathcal{TD}_m^k/(STU, 1T))^*$.

It will be useful later to replace the Lie orientation by an ordering (labeling) of vertices and orientation of each edge. In that case the STU relation is

$$(11) \quad \begin{array}{c} \text{---} \diagup \text{---} \\ \bullet \text{ } j \\ \text{---} \text{---} \text{---} \text{---} \\ \bullet \text{ } i \end{array} = \begin{array}{c} \text{---} \diagup \text{---} \\ \bullet \end{array} - \begin{array}{c} \text{---} \diagdown \text{---} \\ \bullet \end{array}$$

where the planar embedding does not matter.

The following theorem originally appeared for closed knots in D. Thurston's AB Thesis [23]. The extension to invariants of links and a refinement to link-homotopy invariants of string links appears in [14]. This extension treats the case of parallel string links.

Theorem 2.1 ([23, 25, 14]). *The map $\mathcal{W}_m^k \rightarrow \mathcal{V}_m^k/\mathcal{V}_{m-1}^k$ which is inverse to the map w in (9) can be constructed via*

$$W \mapsto \frac{1}{(2m)!} \sum_{\text{labeled } D} W(D) I_D + \text{anomaly term}$$

where the sum is taken over all trivalent diagrams D with labels $\{1, \dots, 2m\}$ on the vertices, and where I_D is a configuration space integral depending on the labeled diagram D .

We define the integral I_D in the next section. The anomaly term is a configuration space integral related to the stratum in configuration space where all points in the configuration have “collided”. Its important feature for our purposes is that it vanishes in the case of the triple linking number for string links. It may not be obvious that the proofs in the existing literature apply to our linearly independent 3-component string links. In addition, the invariant above might depend on a choice of spherical forms used to define the integrals (cf. section 1.2). We will provide the proof for our linearly independent string links in the example of the triple linking number, and we will take note of the appropriate choice of spherical forms (section 3.4). In fact, the simplification in our construction (necessary for Theorem 1) comes precisely from a slightly different proof of this theorem in the case of the triple linking number of linearly independent string links.

Note that $(2m)!$ is precisely the number of labelings of a given isomorphism class of unlabeled (and unoriented) diagrams. Thus the invariant in the Theorem above can equivalently be written as a sum over *unlabeled* diagrams

$$(12) \quad W \mapsto \sum_{\text{unlabeled } D} \frac{1}{|\text{Aut}(D)|} W(D) I_D + \text{anomaly term}$$

where $|\text{Aut}(D)|$ is the size of the group of automorphisms of D . Another way of saying this is that we sum over elements of a basis for \mathcal{TD}_m^k . In particular, each D has an orientation (so that $W(D)$ is well defined), but any given diagram only appears with one Lie orientation in this sum. The formulation in Theorem 2.1 makes the association $D \mapsto I_D$ more immediate, but we find this latter formulation more tractable because there are fewer diagrams to sum over.

2.3. Configuration space integrals. In [5], Bott and Taubes constructed knot invariants by considering a bundle over the space of knots. They worked over the space of closed knots, but we will present the extension of their ideas to spaces of string links. This extension is also discussed in [14]. The bundle is

$$\begin{array}{ccc} F[q_1, \dots, q_k; t] & \longrightarrow & E[q_1, \dots, q_k; t] \\ & & \downarrow \\ & & \mathcal{L}_k = \text{Emb}(\coprod_k \mathbb{R}, \mathbb{R}^3) \end{array}$$

where the fiber $F[q_1, \dots, q_k; t]$ is a compactification of a configuration space of $q_1 + \dots + q_k + t$ points in \mathbb{R}^3 , q_i of which lie on the i^{th} component of the link. Another way of saying this is that the total space $E[q_1, \dots, q_k; t]$ is the pullback in the following square, and the map to \mathcal{L}_k is the left-hand map followed by projection to \mathcal{L}_k .

$$(13) \quad \begin{array}{ccc} E[q_1, \dots, q_k; t] & \longrightarrow & C_{q_1 + \dots + q_k + t}[\mathbb{R}^3] \\ \downarrow & & \downarrow \\ \mathcal{L}_k \times C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right] & \longrightarrow & C_{q_1 + \dots + q_k}[\mathbb{R}^3] \end{array}$$

The pullback square is explained below, and in section 3, we will more carefully examine the fibers $F[q_1, \dots, q_k; t]$ in our particular example.

2.3.1. Brief review of the Axelrod–Singer/Fulton–MacPherson compactification. Here $C_q[M]$ denotes the Axelrod–Singer compactification of the space $C_q(M)$ of configurations of q points on a compact manifold M . It can be defined via blowups. For a submanifold $Y \subset X$, the blowup $\text{Bl}(X, Y)$ is the result of removing Y and replacing it by the sphere bundle of its

normal bundle⁴. Equivalently, this is the result of removing an open tubular neighborhood of Y . Then $C_q[M]$ is defined as the closure of the image of

$$C_q(M) \hookrightarrow M^q \times \prod_{S \subset \{1, \dots, q\} | S| \geq 2} \text{Bl}(M^S, \Delta_S)$$

where Δ_S is the diagonal in M^S . For $M = \mathbb{R}^n$, $C_q[\mathbb{R}^n]$ is considered as the subspace of $C_{q+1}[S^n]$ where the last point is fixed at ∞ . A stratum of $C_q[M]$ is labeled by a collection $\{S_1, \dots, S_k\}$ of distinct subsets $S_i \subset \{1, \dots, q\}$ with $|S_i| \geq 2$ and satisfying the condition

$$S_i \cap S_j \neq \emptyset \Rightarrow \text{either } S_i \subset S_j \text{ or } S_j \subset S_i$$

For each set S_i in the collection, we can think of the points in S_i as having collided. If there is an $S_j \subset S_i$ in the collection, we can think of the points in S_j as having first collided with each other and then with the remaining points in S_i . Two strata indexed by $\{S_1, \dots, S_k\}$ and $\{S'_1, \dots, S'_j\}$ intersect precisely when the set $\{S_1, \dots, S_k, S'_1, \dots, S'_j\}$ satisfies the above condition. In that case, that is the set which indexes the intersection.

Let $s_i = |S_i|$. Roughly speaking, this compactification keeps track of the location of the “collided point” as well as an “infinitesimal configuration” or *screen* for each S_i . Such a screen is a point in the space $(C_{s_i}(T_p M))/G$, where G is the group of translations and (oriented) scalings of Euclidean space. From this one can verify that the stratum labeled $\{S_1, \dots, S_k\}$ has codimension k . Consult [8] and [2] for more precise details.

□

Now it is clear that the right-hand vertical map in the square (13) can be defined as projection to the first $q_1 + \dots + q_k$ points. The lower horizontal map (whose domain is defined below) essentially comes from the fact that an embedding induces a map on configuration spaces which extends to the compactifications. This is somewhat subtle to explain in detail.

2.3.2. A technical digression needed for string links. Let $Q = q_1 + \dots + q_k$, and define $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$ as the closure of the image of the map

$$C_{q_1}(\mathbb{R}) \times \dots \times C_{q_k}(\mathbb{R}) \longrightarrow C_Q[\mathbb{R}^3] (\subset C_{Q+1}[S^3])$$

induced by a fixed string link $L = (L_1, \dots, L_k)$. Then the lower horizontal map in (13) is certainly well defined.

To integrate over the fiber of the bundle $E[q_1, \dots, q_k; t] \rightarrow \mathcal{L}_k$, one needs to know that the closure $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$ has the structure of a manifold with corners. Indeed, it has one induced by that on $C_{q_1 + \dots + q_k + 1}[S^3]$. We sketch some of the details here because this has not been previously explained in the literature. First note that all the added limit points are in

⁴The Fulton–MacPherson compactification in algebraic geometry is defined exactly as the Axelrod–Singer compactification, but with projective blowups instead of real oriented blowups.

the boundary of $C_Q[\mathbb{R}^3]$. Around such a point, a neighborhood in $C_Q[\mathbb{R}^3]$ has various strata, points of which are described by collections of screens, as mentioned above. To describe the corresponding neighborhood in $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$, we replace these spaces of screens by similar spaces which have lower dimension, but will have the same codimension in the latter space:

- At a collision of $s \geq 2$ points at a point p away from ∞ , the space $C_s(T_p\mathbb{R}^3)/G$ is replaced by $C_s(T_pL)/G'$ where by abuse $L = (L_1, \dots, L_k)$ also denotes the image of the fixed string link L and where $G' < G$ is the subgroup of translations and scalings of $T_p\mathbb{R}^3$ which take T_pL to itself.
- Consider a collision of ≥ 1 points with ∞ , first as just a configuration in $C_Q[\mathbb{R}^3] \subset C_{Q+1}[S^3]$. Consider a stratum incident to that configuration, labeled by $\{S_1, \dots, S_k\}$. Let S_i be a set containing the $(Q+1)^{\text{th}}$ point ∞ . We can describe the screen-space corresponding to S_i as $C_{s_i-1}(T_\infty S^3 \setminus \{0\})/\mathbb{R}_+$, by fixing the $(Q+1)^{\text{th}}$ point ∞ at the origin in $T_\infty S^3$.

Now suppose this configuration is also in $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$. We can describe a neighborhood in that space by replacing the screen above by a configuration of points in $T_\infty S^3 \setminus \{0\}$ which are constrained to lie in lines through the origin, modulo scaling. These lines correspond to the directions of the fixed linear embedding. In other words, we replace $C_{s_i-1}(T_\infty S^3 \setminus \{0\})/\mathbb{R}_+$ by $C_{s_i-1}(T_\infty L \setminus \{0\})/\mathbb{R}_+$

Of course, not all the strata in $C_Q[\mathbb{R}^3]$ occur as strata in $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$ because points in different components of the link cannot collide away from ∞ . But for a given $\mathcal{S} = \{S_1, \dots, S_k\}$ which does index a stratum \mathfrak{S} of $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$, any subset of \mathcal{S} clearly indexes a stratum in $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$. These subsets correspond precisely to the higher-dimensional strata which intersect a neighborhood of any point in \mathfrak{S} . This is the sense in which the corner structure on $C_{q_1, \dots, q_k} \left[\coprod_1^k \mathbb{R} \right]$ is inherited from the one on $C_Q[\mathbb{R}^3]$. \square

Remark 2.2 (Notation). In the case of knots, the pullback $E[q_1, \dots, q_k; t]$ above reduces to a space $E[q; t]$ which was called $E_{q,t}$ in our previous work [13], $C_{n,t}$ in Bott–Taubes and Cattaneo–Cotta-Ramusino–Longoni, $C[q+t; \mathcal{K}] = C[q+t; \mathcal{L}_1]$ in [14], and $ev^*C_t(\mathbb{R}^3)$ by Pelatt [21]. We have tried to use the shortest notation that makes clear the difference between this total space and a fiber of the associated bundle.

Now Bott and Taubes showed that in the case of knots, the fiber $F[q; t]$ is a manifold with corners (see their Appendix [5]). Their work together with our section 2.3.2 shows that in general the fiber $F[q_1, \dots, q_k; t]$ is a manifold with corners. This fiber is also orientable. By fixing an orientation on \mathbb{R}^3 and using the orientation of the link, the fibers become *oriented*.

One then integrates differential forms along the fiber of $E[q_1, \dots, q_k; t] \rightarrow \mathcal{L}_k$ and shows that the result is a zero-dimensional closed form, hence a link invariant. Specifically, the

forms are obtained from the maps

$$\varphi_{ij} : C_r(\mathbb{R}^3) \longrightarrow S^2, \quad 1 \leq i < j \leq r$$

given by

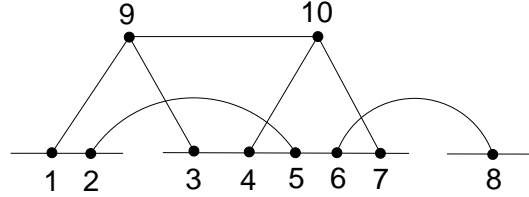
$$\varphi_{ij}(x_1, \dots, x_r) = \frac{x_i - x_j}{|x_i - x_j|}$$

Since the compactification $C_r[\mathbb{R}^3]$ keeps track of directions of collision, these maps extend to $C_r[\mathbb{R}^3]$. Applying this with $r = q_1 + \dots + q_k + t$, we have a map $\varphi_{ij} : E(q_1, \dots, q_k; t) \rightarrow S^2$ for every pair $(i < j)$. Let ω be a 2-form on S^2 representing a generator of $H^2(S^2)$ and let $\theta_{ij} = \varphi_{ij}^* \omega$.

We can now define the I_D in Theorem 2.1. Given a trivalent diagram $D \in \mathcal{TD}_m^k$ with labels $\{1, \dots, 2m\}$ on the vertices, let q_i be the number of vertices on the i th interval, and let t be the number of free vertices. The function I_D is an integral along the fiber $F[q_1, \dots, q_k; t]$ of the bundle $E[q_1, \dots, q_k; t] \rightarrow \mathcal{L}_k$, given by

$$I_D = \int_{F[q_1, \dots, q_k; t]} \theta_{i_1 j_1} \cdots \theta_{i_e j_e}$$

where $(i_1 < j_1), \dots, (i_e < j_e)$ are the pairs of endpoints of edges in D . Since these forms are even-dimensional, their order in the product is irrelevant. For example, to the diagram



we associate the integral

$$\int_{F[2,5,1;2]} \theta_{19} \theta_{25} \theta_{39} \theta_{4,10} \theta_{68} \theta_{79} \theta_{9,10}$$

Since a Lie orientation is equivalent to a vertex-ordering and orientation of every edge, there is a well defined way of associating an integral to an unlabeled Lie-oriented diagram.

It will sometimes be convenient to write $\theta_{i_1 j_1} \cdots \theta_{i_e j_e}$ as

$$\theta_D = \Phi_D^*(\omega_1 \cdots \omega_e)$$

where

$$\Phi_D = \left(\prod_{\text{edges } (i,j) \text{ in } D} \varphi_{ij} \right) : C_{q_1 + \dots + q_k + t}[\mathbb{R}^3] \longrightarrow \prod_{\text{edges } (i,j) \text{ in } D} S^2$$

and ω_i is a 2-form on the i th factor S^2 .

Remark 2.3. In the existing literature, each ω_i is usually rotation-invariant or at least symmetric with respect to the antipodal map. We will proceed slightly differently, using different non-symmetric forms for the ω_i . Thus we will eventually need to keep track of the order of the factors of S^2 in the product above.

The dimension of $F = F[q_1, \dots, q_k; t]$ is $q_1 + \dots + q_k + 3t$, while the dimension of $\theta_{i_1 j_1} \cdots \theta_{i_e j_e}$ is $2e$, where e is the number of edges in D . Since the $q_1 + \dots + q_k$ interval vertices are univalent and the t free vertices are trivalent, $2e = q_1 + \dots + q_k + 3t$, so I_D is a 0-form, i.e., a function.

It is not true in general that I_D is closed, but appropriate linear combinations of various I_D turn out to be closed. The difficulty lies in the fact that the fiber has nonempty boundary. In fact, Stokes' theorem implies that for any form α

$$(14) \quad d \int_F \alpha = \int_F d\alpha \pm \int_{\partial F} (\alpha|_{\partial F E})$$

If α is a linear combination of products of θ_{ij} , then α is closed, so showing that $\int_F \alpha$ is closed amounts to showing that the integral of the restriction of α to $\partial_F E \subset E$ vanishes.

Part of the content of Theorem 2.1 is that $\Sigma_D W(D) I_D$ is closed, i.e., a locally constant function, or link invariant. (In general, this is only true up to the “anomaly term”, though this has been shown to vanish in some cases.) We will review the proof of this part for the particular weight system of interest in section 3.4.

2.4. Milnor triple linking number for string links. Finally, we review the adaptation of the Milnor triple linking number to the case of string links. All we really need is the result about its associated weight system (section 3.2), but we present the definition for completeness. For this section, consider parallel string links. We will relate the two types of string links in section 3.1.

The triple linking number is defined by studying the lower central series of the fundamental group of the link complement. Let L be (the image of) a k -component string link in $I \times D^2$, and let $\pi = \pi_1(I \times D^2 \setminus L)$. Let $\pi^1 = \pi$, and let $\pi^n = [\pi, \pi^{n-1}]$, the n^{th} term in the lower central series. From the Wirtinger presentation of a link, one sees that the meridians and all their conjugates generate π . One can prove (cf. Theorem 4 of [20]) that the meridians generate π/π^n for all n . Thus if F_k denotes the free group on generators a_1, \dots, a_k , there is a map $F_k/F_k^n \rightarrow \pi/\pi^n$ which is surjective, and in fact an isomorphism [20, 10]. Thus there is a word in the a_i which maps to the j^{th} longitude $[\ell_j] \in \pi/\pi^n$.

The *Magnus expansion* is a group homomorphism from F_k to the multiplicative group in the power series ring in noncommuting variables t_1, \dots, t_k ; it is given by $a_i \mapsto 1 + t_i$ and $a_i^{-1} \mapsto 1 - t_i + t_i^2 - \dots$. The *Milnor invariant* $\mu_{i_1, \dots, i_r; j}$ is then defined as the coefficient of $t_{i_1} \cdots t_{i_r}$ in the Magnus expansion of $[\ell_j] \in \pi/\pi^n$. It is straightforward to check that this is well defined for $n > r$.

The invariants $\mu_{i;j}$ are just the pairwise linking numbers. For 3-component string links, there are 6 different “triple linking numbers”, but by permuting labels on the strands it suffices to study just one of them. Thus we take μ_{123} as the triple linking number for string links.⁵

3. THE CONFIGURATION SPACE INTEGRAL FOR THE TRIPLE LINKING NUMBER

In this section, we first use a formula of Mellor to find an expression of the triple linking number for string links in terms of configuration space integrals. Then we examine in detail the compactifications of configuration spaces which appear in these integrals. This will be useful in our description of the invariant as a degree. Then we examine how the integrals combine to make a link invariant. This step is essentially the proof of the isotopy invariance of Theorem 2.1 in the setting of our linearly independent string links, for the triple linking number. The proof is very similar to that in other settings in the existing literature, though ours differs in one place. Going through this proof is also important because it will show us how to glue and collapse boundaries of the configuration spaces in our topological construction. Finally, we check that the Bott–Taubes integrals still give the inverse to the map $\mathcal{V}_m/\mathcal{V}_{m-1} \rightarrow \mathcal{W}_m$ in the setting of linearly independent string links (and with our choices of integrals for them), which may not be a priori obvious. We start with a discussion relating different types of string links.

3.1. Converting between parallel string links and linearly independent string links. Let \mathcal{L}_k^\parallel denote a space of parallel string links, and let \mathcal{L}_k denote string links in some other fixed context allowed under Definition 1.1. We are interested in the case where \mathcal{L}_3 is the space of linearly independent 3-component links. Suppressing the number of components, let \mathcal{V}_m^\parallel denote type- m invariants of \mathcal{L}_k^\parallel , and let \mathcal{V}_m denote type- m invariants of \mathcal{L}_k . A straightening from one type of link to the other (as in Definitions 1.2) induces an isomorphism $\mathcal{V}_m^\parallel \cong \mathcal{V}_m$. The straightening and hence the isomorphism may not be canonical, but each of these vector spaces has a canonical map to \mathcal{W}_m . These are both isomorphisms after quotienting by type- $(m-1)$ invariants, provided Theorem 2.1 also holds in the context of \mathcal{L}_k . In that case we have the following diagram:

$$(15) \quad \begin{array}{ccc} \mathcal{V}_m^\parallel/\mathcal{V}_{m-1}^\parallel & \xrightarrow{\cong} & \mathcal{W}_m \\ \downarrow \text{dotted} & \nearrow \cong & \\ \mathcal{V}_m/\mathcal{V}_{m-1} & & \end{array}$$

⁵Milnor’s original definitions were for closed links, where the μ invariants are only well defined modulo the gcd of the lower order invariants (i.e., pairwise linking numbers in the case of $\mu_{12;3}$). We restrict our attention to string links, where they are well defined integers.

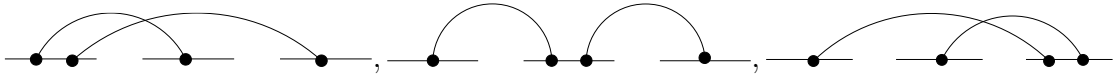
The two isomorphisms shown are canonical maps, and only one choice for the dotted arrow will make the diagram commute. Thus the *quotients* $\mathcal{V}_m^\parallel/\mathcal{V}_{m-1}^\parallel$ and $\mathcal{V}_m/\mathcal{V}_{m-1}$ are canonically isomorphic.

3.2. The weight system for the triple linking number. It is known that for string links, each Milnor invariant $\mu_{i_1 \dots i_r j}$ is finite-type of type r [4, 18]. The triple linking number μ_{123} has an associated weight system W_{123} , given by the map (8). If we consider it as an invariant of our linearly independent string links, it is well defined as an element of $\mathcal{V}_2/\mathcal{V}_1$, but not as an element of \mathcal{V}_2 . Assume for now that Theorem 2.1 holds for linearly independent string links (including the vanishing of all “anomalous terms”). Then its formulation (12) says that

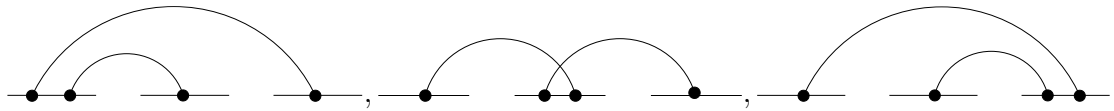
$$\sum_{\text{unlabeled } D} \frac{1}{|\text{Aut}(D)|} W_{123}(D) I_D$$

is an element of \mathcal{V}_2 which in $\mathcal{V}_2/\mathcal{V}_1$ is the class $[\mu_{123}]$ determined by the triple linking number $\mu_{123} \in \mathcal{V}_2^\parallel$ and the canonical isomorphism $\mathcal{V}_2^\parallel/\mathcal{V}_1^\parallel \cong \mathcal{V}_2/\mathcal{V}_1$. So from now until the end of section 3 we will let $\mu_{123} \in \mathcal{V}_2$ denote the right-hand side of the above equality. In checking that Theorem 2.1 holds for linearly independent string links, we will see that for any straightening that induces a certain bijection $\pi_0 \mathcal{L}_3 \cong \pi_0 \mathcal{L}_3^\parallel$, the induced map on link invariants takes our μ_{123} to the bona fide triple linking number, thus excusing our abuse of notation.

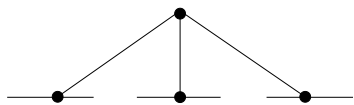
We have $W_{123} \in \mathcal{W}_2^3 \subset (\mathcal{CD}_2^3)^*$, a functional on diagrams with four vertices. Since the triple linking number is an invariant of link-homotopy, we know that W_{123} can be nonzero only on diagrams where no chord joins vertices on the same strand [14]. We may furthermore consider only diagrams where each of the three strands has at least one vertex, since the triple linking number cannot detect crossing changes between only two components. There are seven such diagrams spanning⁶ a subspace of \mathcal{CD}_2^3 : the diagrams L, M, R



the diagrams L', M', R'



and the diagram T



⁶These seven diagrams satisfy relations in $\mathcal{CD}_2^3/(4T)$, but the calculation below is easy enough that we do not need to use these relations.

Since the strands are labeled and oriented, each of these diagrams has no nontrivial automorphisms. Thus we have so far

$$\mu_{123} = \sum_{D \in \{L, M, R, L', M', R', T\}} W_{123}(D) I_D.$$

Using a skein relation for the Milnor invariants proven by Polyak [22], Mellor proved a recursive formula for the weight systems of these invariants. This formula is Theorem 2 of [18], to which the reader may refer for details. We apply Mellor's formula to the weight system W_{123} and the six chord diagrams L, M, R, L', M', R' . The result is that W_{123} vanishes on L', M' , and R' , and its values on L, M, R are $1, -1, 1$.

The STU relation will ultimately tell us the value of W_{123} on T . But first, to clarify the association $D \mapsto I_D$ in this setting of unlabeled diagrams, the canonical Lie orientation on a chord diagram corresponds to a canonical ordering of the vertices (from left to right) and a canonical orientation of the edges (also from left to right). This defines I_D for the chord diagrams D . For the diagram T , we can choose any vertex-ordering and edge-orientations and then account for the sign by using the labeled STU relation (11) to compute $W_{123}(T)$. We choose to label the free vertex “4” and orient the edges from smaller to larger index.

Remark 3.1. If one considers an unlabeled STU relation 10 featuring the unlabeled, Lie-oriented basis element T , one gets the same sign for $W_{123}(T)$ as the one we get for the labeling we chose. Thus under the correspondence between the two types of orientation, our choice of labeling of T gives the same orientation as the unlabeled, Lie-oriented basis element T .

The calculation described above yields the following result, modulo checking that Theorem 2.1 extends to linearly independent 3-component string links.

Corollary 3.2 (to Theorem 2 of [18] and Theorem 2.1). *The triple linking number for string links is, up to a type-1 invariant, the sum of integrals*

$$\mu_{123} = I_L - I_M + I_R - I_T$$

where the terms I_D above are the integrals over compactified configuration spaces associated to the diagrams $D = L, M, R, T$:

$$\begin{aligned} I_L &= \int_{F[2,1,1;0]} \theta_{13} \theta_{24}, & I_M &= \int_{F[1,2,1;0]} \theta_{12} \theta_{34} \\ I_R &= \int_{F[1,1,2;0]} \theta_{13} \theta_{24}, & I_T &= \int_{F[1,1,1;1]} \theta_{14} \theta_{24} \theta_{34} \end{aligned}$$

We can remove the “up to a type-1 invariant” indeterminacy by making appropriate choices in defining these integrals.

3.3. The compactified configuration spaces. In our topological construction, it will be useful to understand in detail the compactifications of these four configuration spaces, which

we will sometimes abbreviate as F_L, F_M, F_R, F_T .

3.3.1. The space F_T . First we describe $F_T = F[1, 1, 1; 1]$, the fiber of the bundle $E[1, 1, 1; 1]$ over a 3-component string link L . This space compactifies $F(1, 1, 1; 1) = \{(x_1, \dots, x_4) \in C_4(\mathbb{R}^3) | x_1, x_2, x_3 \in \text{im } L\}$. By forgetting x_4 which is free to roam in \mathbb{R}^3 , the uncompactified $F(1, 1, 1; 1)$ maps to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as a fiber bundle with fiber homeomorphic to $\mathbb{R}^3 \setminus \{3 \text{ } D^3\text{'s}\}$, the complement of 3 closed 3-balls in \mathbb{R}^3 . The base is contractible, so $F(1, 1, 1; 1) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^3 \setminus \{3 \text{ } D^3\text{'s}\})$.

The compactification $F[1, 1, 1; 1]$ is a subset of the Axelrod–Singer compactification $C_4[\mathbb{R}^3] \subset C_5[S^3]$. Since $F[1, 1, 1; 1]$ is a manifold with corners, it has a collar neighborhood of the boundary and is homotopy-equivalent to its interior, $F(1, 1, 1; 1)$. In $F[1, 1, 1; 1]$ the subsets which index codimension-1 faces are

$$\{x_1, x_4\}, \{x_2, x_4\}, \{x_3, x_4\}$$

as well as the 15 sets

$$\{x_1, \infty\}, \{x_2, \infty\}, \{x_3, \infty\}, \{x_4, \infty\}, \dots, \{x_1, x_2, x_3, x_4, \infty\}$$

consisting of a nonempty subset of the x_i together with ∞ . If S is a set of the second type, the corresponding face has $2^{a(S)}$ connected components, where $a(S) = \#(S \cap \{x_1, x_2, x_3\})$.

For any of these compactified configuration spaces, we call a face whose indexing set contains precisely two x_i a *principal face*. We call a face whose indexing set contains the symbol ∞ a *face at infinity* or, for reduced verbiage, just an *infinite face*. In all the spaces which we consider here, every face is either principal or infinite.

As most of the faces here are infinite, it seems useful to explicitly describe an infinite face. In $C_n[\mathbb{R}^3]$, the face where m of the n points have escaped to infinity can be described as the product $C_{n-m}(\mathbb{R}^3) \times (C_{m+1}(\mathbb{R}^3)/G)$ where G is the group of translations and scalings of \mathbb{R}^3 (cf. section 6, p. 18 of [24]). The $(m+1)^{\text{th}}$ point in the second factor is a “fat point” of points which have not escaped to infinity. If we translate this point to the origin, then we can rewrite such a face as $C_{n-m}(\mathbb{R}^3) \times (C_m(\mathbb{R}^3 \setminus \{0\})/\text{scaling})$. (Alternatively, $\mathbb{R}^3 \setminus \{0\}$ could be replaced by the complement of a ball around the origin.) Note that this description applies even when $m = n$. In $F[1, 1, 1; 1]$ an infinite face has a similar description, except that the x_i are constrained to lie on the fixed linear embedding.

3.3.2. The spaces F_L, F_M, F_R . Now we describe the fiber $F_L = F[2, 1, 1; 0]$ of the bundle $E[2, 1, 1; 0]$, which compactifies $F(2, 1, 1; 0) = \{(x_1, \dots, x_4) \in C_4(\mathbb{R} \sqcup \mathbb{R} \sqcup \mathbb{R}) | (x_1 < x_2) \in C_2(\mathbb{R}), x_3 \in \mathbb{R}, x_4 \in \mathbb{R}\}$. The latter space is homeomorphic to an open 4-ball. As in the case of F_T above, the compactification has a collar neighborhood of the boundary, and is homotopy-equivalent to its interior $F(2, 1, 1; 0)$.

The subsets which index codimension-1 faces of $F[2, 1, 1; 0]$ are $\{x_1, x_2\}$ and the 15 sets consisting of a nonempty subset of $\{x_1, \dots, x_4\}$ together with ∞ . The compactification $F[2, 1, 1; 0]$ is a manifold with corners, but topologically it is homeomorphic to a closed ball. As with F_T , an infinite face may have multiple components, and the description of infinite faces in the previous section apply equally well here.

The spaces $F_M = F[1, 2, 1; 0]$ and $F_R = F[1, 1, 2; 0]$ are constructed in a similar manner and are diffeomorphic to $F_L = F[2, 1, 1; 0]$.

3.4. Vanishing Arguments. Having examined the compactifications in our example, we now examine how the configuration space integrals combine to form a link invariant. The first important point is that the forms we integrate extend to the compactifications.

Lemma 3.3. *For each diagram $D \in \{L, M, R, T\}$, there is a smooth map*

$$\Phi_D : F_D \longrightarrow \left(\prod_{\text{edges } (i,j) \text{ in } D} S^2 \right)$$

extending the map $(x_1, \dots, x_4) \mapsto \prod (x_i - x_j)/|x_i - x_j|$ from the interior of F_D to all of F_D .

Proof. The compactifications of the F_D are gotten by blowing up every diagonal, so the direction of collision of x_i with x_j is recorded. The only case which might have been unclear (and possibly overlooked in some existing literature) is that of the (many!) infinite faces. From their description at the end of section 3.3.1, one sees that Φ_D is well defined on them too. \square

Next, a configuration space integral is only a link invariant because the integrals along the boundary faces vanish or cancel each other. (The necessity of this condition comes from Stokes' Theorem, cf. equation (14).) The next lemmas are important because they indicate how we should collapse and glue together configuration spaces. We give an original proof of Lemma 3.5 which relies on the behavior of our string links towards infinity and which then simplifies the gluing constructions.

Lemma 3.4. *The integrals along the three principal faces of F_T cancel with those along the three principal faces of F_L, F_M , and F_R .*

Lemma 3.5. *The integral along any infinite face of F_D vanishes, where D is any of L, M, R, T .*

Proof of Lemma 3.4. This proof is given in [23, 25, 14], and the main idea is to use the STU relation and the faces corresponding to the terms in the relation (and the restrictions of the forms θ_D to those faces) are identical, up to a factor of S^2 . In this case, Mellor's formula says that W_{123} vanishes on L', M', R' . Thus, with our vertex-orderings and edge-orientations

on L, M, R, T , the STU relations are

$$W_{123}(T) = -W_{123}(L) \quad W_{123}(T) = W_{123}(M) \quad W_{123}(T) = -W_{123}(R).$$

Each of the three principal faces of F_T is diffeomorphic to $S^2 \times F[1, 1, 1; 0]$, and the restriction of the form to be integrated is $\omega \theta_{ij} \theta_{ik}$ (where i, j, k are distinct elements of $\{1, 2, 3\}$ and where ω is the 2-form on the first factor S^2). Each of the three principal faces of F_L, F_M, F_R is diffeomorphic to $F[1, 1, 1; 0]$, and the restriction of the form to be integrated is $\theta_{ij} \theta_{ik}$ (with i, j, k as before). Since ω integrates to 1, the integral along a principal face of F_T is, up to sign, the integral along a principal face of F_L, F_M, F_R . One observes that signs coming from the restriction of the forms, the orientations on the faces of the F_D , and the coefficients $W_{123}(D)$ are such that each of the three pairs of integrals along principal faces cancels. We will give more explicit details when we recast this topologically in sections 4.3 and 5. \square

Proof of Lemma 3.5. The above references suggest the main idea of our proof, which is that the image of each infinite face under Φ_D is a proper subspace of $S^2 \times S^2 \times S^2$. This allows us to choose a form on $S^2 \times S^2 \times S^2$ which generates its top (integral) cohomology, but which is supported on the complement of the image of Φ_D . Then θ_D , its pullback via Φ_D , is zero. If the image of Φ_D had codimension ≥ 2 (as in the case of knots), any choice of form would suffice. In our setting of linearly independent string links, our choice of form matters, but a correct choice will produce all the needed results. Below are details for all the cases:

- (a): Suppose \mathfrak{S} is an infinite face of E_D such that the indexing set S does not contain a free vertex. Let $x_i \in S$ and let x_j be the point connected to x_i by a chord. Then the image $\varphi_{ij}(\mathfrak{S})$ is contained in a positive-codimension subspace C of S^2 . In particular, if \vec{v}_i and \vec{v}_j are the directions of the fixed linear maps for the strands containing x_i and x_j , then C is the great circle which is the intersection of $\text{span}\{\vec{v}_i, \vec{v}_j\}$ with the unit sphere. We will from now on take our fixed linear maps to be the x -, y -, and z -axes, in which case the image of any face under any φ_{ij} is contained in the union of the three equators determined by the coordinate planes.
- (b): Now suppose the indexing set S of \mathfrak{S} contains a free vertex. In other words \mathfrak{S} is an infinite face of E_D with $D = T$, and $x_4 \in S$. If two (of the remaining three) points x_i, x_j are *not* in S , then $\varphi_{i4} \times \varphi_{j4}(\mathfrak{S})$ is contained in the diagonal of $S^2 \times S^2$. So suppose that there are two other points $x_i, x_j \in S$. Since these two points must be on the link, these points lie in a plane, by the fixed behavior of the link towards infinity.

As in case (a), suppose that the three fixed linear maps $\mathbb{R} \hookrightarrow \mathbb{R}^3$ for our string link are the coordinate axes. Choose the three 2-forms $\omega_1, \omega_2, \omega_3$ on the factors of $S^2 \times S^2 \times S^2$ to have support in $Q_1 := \{y > 0 \text{ and } z < 0\}$, $Q_2 := \{z > 0 \text{ and } x < 0\}$, and $Q_3 := \{y < 0 \text{ and } x > 0\}$ respectively, where we consider each S^2 as the unit sphere in xyz -space. (In addition, they should still vanish on all the codimension-1

subsets C from case (a).) Then the product $\omega_i \omega_j$ (where $i \neq j$ are any two of the indices 1,2,3) has support in $Q_i \times Q_j \subset S^2 \times S^2$. But the restriction of $\varphi_{i4} \times \varphi_{j4}$ to this infinite face \mathfrak{S} is contained in the complement of $Q_i \times Q_j$. Hence the pullback of $\omega_i \omega_j$ is zero. (A similar argument with obvious modifications applies when the three fixed directions of the string link components towards infinity are arbitrary linearly independent directions. A judicious choice of the 2-forms on S^2 is crucial for this argument.)

□

Remark 3.6. From here on, we will continue to take the fixed linear map to be three axes, and the forms $\omega_1, \omega_2, \omega_3$ as above. Note that each of these forms is neither symmetric with respect to the antipodal map nor symmetric with respect to any rotation. Since we will glue the four configuration spaces together, we will have to use forms ω_i as above for all four integrals (section 5), even though case (b) above only concerned the space E_T .

Remark 3.7. For the particular diagrams we consider, we could also argue the vanishing in case (b) using an involution. The involution is the same as the one of Kontsevich [12] (see also [5], [23] [25]), except in the case where all points approach infinity, where it is the antipodal map. We have chosen to use non-symmetric forms as above because the resulting gluing construction is simpler: this method requires only working relative to boundary faces, rather than “folding” the faces along which the integrals vanish according to the involution. This simplification will make clear that we have a map from S^6 to the fiber of our glued space modulo its boundary, which induces an isomorphism in 6-dimensional (co)homology. This leads to our description of the invariant as a degree. If we were using linearly independent string links, we could not use the same argument in case (b) above, and we would have had to resort to involutions, thus hindering our degree description.

3.5. Universality of Bott–Taubes integration for linearly independent string links.

We have examined how μ_{123} , as defined for our linearly independent string links, is an isotopy invariant. It remains to check that in our setting it gives the correct invariant, i.e., that the integration map $\mathcal{W}_m \rightarrow \mathcal{V}_m/\mathcal{V}_{m-1}$, which takes $W_{123} \mapsto \mu_{123}$, is inverse to the canonical map $\mathcal{V}_m/\mathcal{V}_{m-1} \rightarrow \mathcal{W}_m$, at least on the weight system W_{123} . The proof of the following lemma is similar to the proof of Theorem 5.8 in [14], which uses the idea of “tinkertoy diagrams” of D. Thurston [23]. In our setting we cannot take the spherical forms to be symmetric or all have the same support, and we cannot make our links “almost planar”. So a priori it may not be obvious that the arguments in those references apply in our setting.

Lemma 3.8. *The map $\mathcal{V}_2/\mathcal{V}_1 \rightarrow \mathcal{W}_2$ takes our μ_{123} to the weight system W_{123} coming from the triple linking number.*

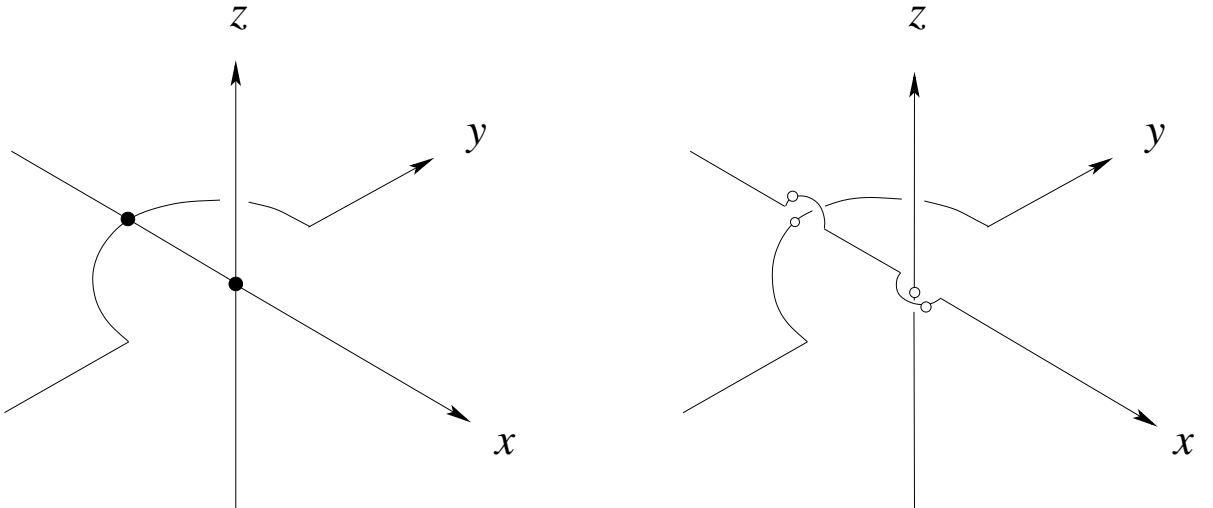
Sketch of proof: We proceed by choosing for each chord diagram $D \in \mathcal{CD}_2^3$ a singular link K_D realizing the singularities prescribed by D . Then we show that $\mu_{123}(K_D)$ (defined using the Vassiliev skein relation) is equal to $W_{123}(D)$.

Start by requiring $\omega_1, \omega_2, \omega_3$ to have support in a neighborhood of radius δ of the points $\mathbf{v}_1 = u((\gamma, 1, -1))$, $\mathbf{v}_2 = u((-1, \gamma, 1))$, $\mathbf{v}_3 = u((1, -1, \gamma))$ respectively, where $u(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$. Here γ is a nonzero number such that the neighborhood of radius δ of each \mathbf{v}_i is disjoint from all the coordinate planes. We could take $\gamma = 1$, but thinking about γ small might make our tinkertoy argument clearer. Note that the support of ω_i can have satisfy these conditions, while still satisfying all of those given in case (a) and (b) in the proof of Lemma 3.5. The tinkertoy idea is to count configurations where we can put rods between vertices which point in the directions of these vectors.

Step 1: $\mu_{123}(K_D) = W_{123}(D)$ for $D = L, M, R, L', M', R'$.

For the diagrams $D = L, M, R, L', M', R'$ we choose K_D as follows. For $D = L$ for example, take one of the strands with one vertex, say the y -axis, and send it in a big semicircle in the xy -plane around the other one-vertex strand, the z -axis. The semicircle goes into the negative x -half-space because y hits x before z does in the diagram L . Leave the other two strands as the axes. This leaves us with two double points, as shown below. By the skein relation, we replace K_D by a signed sum of four nonsingular links, obtained by resolving the two double points (in the $2 \cdot 2 = 4$ possible ways), inside a small ball of radius ε around each singularity.

FIGURE 4. On the left is a singular link K_D for $D = L$. On the right is one of the four resolutions of K_L together with a configuration of 4 points of the type specified by L . It is the only resolution on which the integral is nonzero, and the configuration shown is the (only) one that the integral counts. Both resolutions are “negative” in the sense of the Vassiliev skein relation, so the contribution is $+1$.



For any chord diagram $\tilde{D} \in \mathcal{CD}_2^3$, the integral $I_{\tilde{D}}$ counts configurations of the two pairs of points (joined by a chord) such that each pair's difference unit vector is equal to the appropriate \mathbf{v}_i and such that the points are on the strands in the order specified by \tilde{D} . The reader can verify that by our choice of \mathbf{v}_i and choice of links, the only cases where both pairs of points are in a position to contribute towards $I_{\tilde{D}}(K_D)$ are the following:

- Each pair is near a resolved singularity, as in the right-hand picture above. Such a configuration only contributes if the points are in the correct order, which only happens if $D = \tilde{D}$. Moreover, in this case, a contributing configuration only occurs for one of the two resolutions at each double point. The reader can verify that with our choices of quadrants Q_1, Q_2, Q_3 , the signs of the two contributing resolutions are always the same for $D = L, M, R$. So this gives a contribution of $+1$ to $I_D(K_D)$.
- One pair is near a resolved singularity, while another consists of a point on the large semicircle and a point on an axis away from the singularities. In this case, the contribution is the same for the two resolutions of the singularity away from both pairs. Thus two of the four terms are zero, and the other two cancel each other.

For the integral I_T , we count configurations where the unit vector difference between the free vertex and the three interval vertices is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ respectively. The reader can verify that any two of the three constraints imply that the free vertex is near a singularity, but then the third constraint is impossible to satisfy.

Step 2: If D has a chord joining points on the same strand, $\mu_{123}(K_D) = W_{123}(D) = 0$.

In this case, we can just choose any K_D where the double point(s) on the same strand are sufficiently far away from the other strands. The integrals I_L, I_M, I_R, I_T do not count rods between two points on the same strand, so their value on the two resolutions of a double point on one strand will be ε -close. Since we can choose ε arbitrarily small while staying in the same isotopy class, the integrals on these two resolutions will cancel.

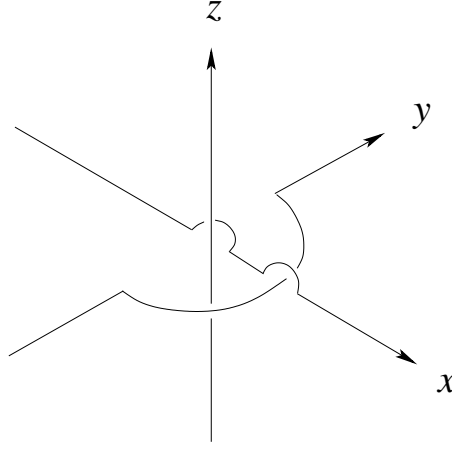
Step 3: $\mu_{123}(K_D) = W_{123}(D)$ for all D .

The remaining case is chord diagrams with two chords on two strands, with no vertices on the remaining strand. In this case take K_D again to be a singular link where the strand i with no vertices follows a large semicircle around one of the other two strands. Near the apex of the semicircle, the picture will look like the right-hand side of the Figure above, while the singularities will be near the origin. Each of I_L, I_M, I_R counts configurations involving the strand i , so any nonzero contributions will cancel in pairs, since all the resolutions involve only the other two strands. By keeping the singularities sufficiently close to the origin (with respect to the radius of the large semicircle), we see that I_T vanishes on such a K_D for the same reason it vanished on the K_D in Step 1. \square

Our choice to use linearly independent string links had resulted in some gaps in the proof of Corollary 3.2, but these are now nearly all filled. We finish by removing the “up to a type-1 invariant” indeterminacy mentioned in Corollary 3.2.

First, we observe that the tinkertoy method of proof above shows that all four of the integrals in μ_{123} vanish on the link shown in the figure below. We designate this link U

FIGURE 5. A good candidate for the unlink in our setting, given the choices we have made.



to be the unlink among our linearly independent string links. Passing strand i through strand j produces a link L_{ij} which must correspond to a parallel string link whose linking number of strands i and j is ± 1 . This is because any choice of linking number will have different values on U and L_{ij} . One can verify that all four integrals in μ_{123} vanish on any L_{ij} . Lemma 3.8 showed that μ_{123} agrees with the triple linking number up to a type-1 invariant v_1 (cf. diagram (15)). It is easily shown that a type-1 invariant is a sum of pairwise linking numbers and a constant. Since μ_{123} vanishes on the unlink, the constant term in v_1 must be zero. Since μ_{123} vanishes on every L_{ij} , the coefficient of every pairwise linking number must be zero. Hence using any straightening from linearly independent string links to parallel string links which induces a bijection as above, μ_{123} is the triple linking number on the nose, validating Corollary 3.2.

Remark 3.9. We can explain our choice of unlink seem in a way that seems less ad hoc: applying any number (i.e., 0,1,2,3) of the three crossing changes indexed by $\{x, y\}$, $\{y, z\}$, $\{x, z\}$ to our link U gives a 3-dimensional cube of links. Each edge connects links that differ by one crossing change. If we relaxed the behavior towards infinity to *affine-linear*, each of these eight links could be drawn with all strands as straight lines. These eight links are the four resolutions of K_L and $K_{L'}$. They are also the resolutions of K_M and $K_{M'}$, with the two groups of four grouped differently. Similarly we could replace L or M by R . On exactly three of these, $\mu_{123} = 1$ because $I_D = 1$ for exactly one $D \in \{L, M, R\}$. The cube can be

positioned so that these three links are at the $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ vertices. Then U is at the $(1, 1, 1)$ vertex, adjacent to three links with vanishing μ_{123} . But each pairwise linking number must take different values on U and one of these three adjacent links.

4. GLUING MANIFOLDS WITH FACES

Lemmas 3.4 and 3.5 show that we should glue our bundles along their principal faces and work relative to, or collapse, the infinite faces. To prove Theorem 1, we need the glued space to be a manifold with corners. To embed it with a tubular neighborhood and normal bundle and thus prove Theorem 2, we need it to be a nice type of manifold with corners, namely a manifold with faces. We first recall some definitions related to manifolds with faces. Next we prove some general statements which will be useful in understanding how to do a Pontrjagin–Thom construction with our glued space. Then we construct a glued space E_g , and, as the main result of this section, we show that E_g is a manifold with faces.

4.1. Gluing manifolds with faces. We start by reviewing some basic definitions. The notion of manifolds with faces and $\langle N \rangle$ -manifolds goes back to Jänich [11], but here, as in our previous work [13], we follow the work of Laures [16]. First recall that in a *manifold with corners* X every $x \in X$ has a neighborhood diffeomorphic to a neighborhood of the origin in $\mathbb{R}^{n-k} \times [0, \infty)^k$. Call the number k the *codimension* $c(x)$ of the point x . Let C_k be the set of points in X of codimension k . Call the closure of a connected component of C_1 a *connected face*. We say that X is a *manifold with faces* if X is a manifold with corners such that for any $x \in X$, x is contained in $c(x)$ different connected faces of X . A $\langle N \rangle$ -*manifold* is a manifold with faces X together with a decomposition of the boundary $\partial X = \partial_1 X \cup \dots \cup \partial_N X$ such that each $\partial_i X$ is a disjoint union of connected faces and such that $\partial_i X \cap \partial_j X$ is a disjoint union of connected faces of each of $\partial_i X$ and $\partial_j X$.

It will be convenient to know the following basic lemma.

Lemma 4.1. *Any manifold with faces X can be given the structure of a $\langle N \rangle$ -manifold for some N .*

Proof. Let $\partial_1 X, \dots, \partial_N X$ be the connected codimension-1 faces of X . To verify that $\bigcup_i \partial_i X = \partial X$, we have to check that for any $x \in \partial X$, every neighborhood of x contains points of codimension 1. But this follows immediately from X being a manifold with corners.

We also have to check that the intersection $\mathcal{I} := \partial_i X \cap \partial_j X$ is a disjoint union of connected faces of each of $\partial_i X$ and $\partial_j X$. Note that points of codimension 1 in $\partial_i X$ or $\partial_j X$ are codimension-2 in X . Given $x \in \mathcal{I}$, consider a chart from a neighborhood U of x in X to a ball B around the origin in some $\mathbb{R}^{n-k} \times [0, \infty)^k$. This must take $U \cap \partial_i X$ and $U \cap \partial_j X$ to two codimension-1 faces of B , and the intersection of these contains a codimension-2 subset

of B . Since the above holds for any U , x must be in the closure of the set of codimension-2 points C_2 . That is, $\mathcal{I} \subset \overline{C_2}$.

To see that \mathcal{I} contains the closure of every connected component of C_2 which it intersects, suppose $x \in \mathcal{I}$ and suppose $y \in C_2$ such that there is a path in C_2 from x to y . Covering this path with neighborhoods in $C_0 \cup C_1 \cup C_2$, we see that points of C_1 near y are in the same component of C_1 as points of C_1 near x . Since $\partial_i X$ and $\partial_j X$ are closures of components of C_1 , $y \in \mathcal{I}$.

Finally, we just need to check that the closures of components of C_2 contained in \mathcal{I} are disjoint. Suppose we have two components C, C' of C_2 whose closures have nonempty intersection. Then their intersection must contain a point z of codimension ≥ 3 . A neighborhood of z in X is modeled by a ball B around the origin in $\mathbb{R}^{n-k} \times [0, \infty)^k$ with $k \geq 3$. Under the coordinate map, two distinct codimension faces in X go to codimension-1 faces of B . But the intersection of two such faces is a single connected codimension-2 subset. Hence C and C' cannot both be in the intersection $\mathcal{I} = \partial_i X \cap \partial_j X$. \square

We continue discussing glued spaces in general terms.

Lemma 4.2. *Let X_1 and X_2 be two manifolds with faces. Suppose $\mathfrak{S}_i \subset \partial X_i$ for $i = 1, 2$ are codimension-1 faces of X_i and are diffeomorphic as manifolds with corners via an orientation-reversing diffeomorphism $g : \mathfrak{S}_1 \cong \mathfrak{S}_2$. Then we can form a new manifold with faces $X = X_1 \cup_g X_2$ by gluing the X_i via g .*

Proof. We first check that X is a manifold with corners. Let $x \in \mathfrak{S}_i$ be a point with codimension $c(x) = k$. Then in each X_i (using the diffeomorphism g), x has a neighborhood diffeomorphic to a “ball” around the origin in $\mathbb{R}^{n-k} \times [0, \infty)^k$. If we glue two such balls along the codimension-1 faces which map to the \mathfrak{S}_i , then the result can obviously be identified with a “ball” in $\mathbb{R}^{n-k+1} \times [0, \infty)^{k-1}$. We use this to give X the structure of a smooth manifold with corners. Note that a point in \mathfrak{S}_i that was codimension k in X_i now has codimension $k - 1$ in X .

To give X the structure of a manifold with faces, we glue all the strata which intersect \mathfrak{S}_i in pairs (which can be thought of as “mirror images” across \mathfrak{S}_i). Thus the (codimension-1) connected faces of X consist of those connected faces of X_i which don’t intersect \mathfrak{S}_i , together with one connected face for every element in the set

$$\{(V_1, V_2) \mid V_i \text{ a (codim-1) conn. face of } X_i, V_i \cap \mathfrak{S}_i \neq \emptyset, V_i \neq \mathfrak{S}_i, g(V_1 \cap \mathfrak{S}_1) = V_2 \cap \mathfrak{S}_2\}.$$

Suppose a point $x \in \mathfrak{S}_i$ had codimension k in (either) X_i . Then it was contained in k faces of X_i . In X , it is contained in $k - 1$ faces: one for each of the k faces except \mathfrak{S}_i itself. But as noted above, x has codimension $k - 1$ in X . The condition is obviously satisfied for all $x \notin \mathfrak{S}_i$. \square

By the lemma 4.1, the glued-up space X is a $\langle N \rangle$ -manifold for some N . The content of Proposition 2.1.7 of Laures' paper [16] is that compact $\langle N \rangle$ -manifolds are precisely the compact spaces which admit *neat* embeddings into Euclidean space with corners $\mathbb{R}^M \times [0, \infty)^\mathbb{N}$. For a precise definition of a neat embedding, see Laures' paper ([16], Definition 2.1.4; also repeated in the author's previous work [13] as Definition 3.1.2). For our construction, the important features are (1) that neat embeddings have well defined normal bundles and (2) that the manifold being embedded meets all the boundary strata of $\mathbb{R}^M \times [0, \infty)^\mathbb{N}$ perpendicularly.

Corollary 4.3. *Let X_1 and X_2 be two compact manifolds with faces. Suppose $\mathfrak{S}_i \subset \partial X_i$ and $g : \mathfrak{S}_1 \cong \mathfrak{S}_2$ are as above. Then $X = X_1 \cup_g X_2$ can be neatly embedded into some Euclidean space with corners.*

Finally note how the construction in Lemma 4.2 can be repeated inductively. That is, suppose X_1, \dots, X_k are manifolds with faces, and $\mathfrak{S}_1, \mathfrak{S}'_1, \dots, \mathfrak{S}_m, \mathfrak{S}'_m$ are codimension-1 faces in $X_{i_1}, X_{j_1}, \dots, X_{i_m}, X_{j_m}$ such that all the \mathfrak{S}_ℓ and \mathfrak{S}'_ℓ are pairwise disjoint and such that there are diffeomorphisms $g_\ell : \mathfrak{S}_\ell \cong \mathfrak{S}'_\ell$ for each ℓ . Then we can form a glued space $X_1 \sqcup \dots \sqcup X_k / \sim$, where the quotient is determined by g_1, \dots, g_m . From the proof of Lemma 4.2, we see that this glued space is again a manifold with faces, hence an $\langle N \rangle$ -manifold for some N .

4.2. Fiberwise gluings of manifolds with faces. Now let $F_1 \longrightarrow E_1 \longrightarrow B$ and $F_2 \longrightarrow E_2 \longrightarrow B$ be fiber bundles where each F_i is a compact manifold with faces. Let $F_i(b)$ denote the fiber of E_i over $b \in B$. Suppose that (for $i = 1, 2$) we have sub-bundles $\mathfrak{S}_i \subset E_i \rightarrow B$ where the fiber $F(\mathfrak{S}_i)(b)$ over each $b \in B$ is a codimension-1 face of $F_i(b)$. Suppose that for each $b \in B$, $F(\mathfrak{S}_1)(b), F(\mathfrak{S}_2)(b)$ are diffeomorphic as manifolds with corners via an orientation-reversing diffeomorphism $g_b : F(\mathfrak{S}_1)(b) \cong F(\mathfrak{S}_2)(b)$, and suppose further that these diffeomorphisms vary smoothly over B . (That is, fix one copy $F(\mathfrak{S})$ of $F(\mathfrak{S}_i)(b)$; then given a trivialization of E_i over $U \subset B$ as $U \times F_i$, the induced map from $U \times F(\mathfrak{S}_i)$ to this fixed $F(\mathfrak{S})$ is smooth.) Then we can fiberwise glue E_1 to E_2 along $F(\mathfrak{S}_i)$ to form a bundle $F_g \longrightarrow E_g \longrightarrow B$ whose fibers $F_g = F_1 \cup_g F_2$ are manifolds with faces, or $\langle N \rangle$ -manifolds for some N . Similarly, we can carry out a fiberwise analogue of the construction at the end of the previous paragraph with more than two bundles E_1, \dots, E_k , provided that the faces along which we glue are pairwise disjoint.

4.2.1. Fiberwise integration on glued manifolds with corners. Suppose that, as above, we glue bundles $F_i \longrightarrow E_i \longrightarrow B$ ($i = 1, 2$) to a bundle $F_g \longrightarrow E_g \longrightarrow B$. Suppose α_i for $i = 1, 2$ are forms on E_i such that the forms $g^*(\alpha_2)$ and α_1 agree on \mathfrak{S}_1 . The forms α_i can then be glued to a form α on E_g which is at least piecewise smooth. Without any further assumptions, we do not have fiber integration on the level of cohomology for the bundles $E_i \rightarrow B$. But on the level of forms, this much is obvious:

Lemma 4.4. *In the above situation,*

$$\int_{F_1} \alpha_1 + \int_{F_2} \alpha_2 = \int_{F_1 \cup_g F_2} \alpha.$$

A similar statement holds in the case of gluing bundles E_1, \dots, E_k with $k > 2$. \square

Since integration can be done piecewise, the manifold with faces structure (or even the manifold with corners structure) on E_g is not strictly necessary for this lemma. But the particular glued space E_g which we will construct will anyway be a manifold with faces.

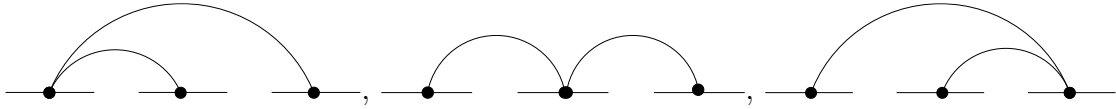
4.3. Gluing the bundles over the link space. Now we construct our particular glued space E_g , which is a bundle over \mathcal{L}_3 .

The spaces E_D ($D = T, L, M, R$) have precisely the corner structure of their fibers and hence principal faces corresponding to those in the fibers. Let $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ denote the principal faces of $E_T = E[1, 1, 1; 1]$, and let $\mathfrak{S}_L, \mathfrak{S}_M, \mathfrak{S}_R$ denote the principal faces of $E_L = E[2, 1, 1; 0]$, $E_M = E[1, 2, 1; 0]$, $E_R = E[1, 1, 2; 0]$ respectively. We denote the fibers of these faces (or faces of the fibers) $F(\mathfrak{S}_i)$, $i = 1, 2, 3$ and $F(\mathfrak{S}_D)$, $D = L, M, R$.

We observed in section 3.3 that each of $F(\mathfrak{S}_1), F(\mathfrak{S}_2), F(\mathfrak{S}_3)$ is diffeomorphic to $F[1, 1, 1; 0] \times S^2$, and that each of $F(\mathfrak{S}_L), F(\mathfrak{S}_M), F(\mathfrak{S}_R)$ is diffeomorphic to $F[1, 1, 1; 0]$. Moreover, we have an orientation-preserving diffeomorphism from $F(\mathfrak{S}_i)$ ($i = 1, 2, 3$) to $F(\mathfrak{S}_D) \times S^2$ ($D = L, M, R$) where the orientation on S^2 comes from the one we fixed on \mathbb{R}^3 .

We glue F_L, F_M , and F_R to F_T by first reversing the orientation on F_T , and then identifying $F(\mathfrak{S}_L) \times S^2$ with $F(\mathfrak{S}_1)$, $F(\mathfrak{S}_M) \times S^2$ with $F(\mathfrak{S}_2)$, and $F(\mathfrak{S}_R) \times S^2$ with $F(\mathfrak{S}_3)$. Note that we have a specific (orientation-reversing) diffeomorphism over each fiber, and these diffeomorphisms vary smoothly over \mathcal{L}_3 . Thus the gluing of the fibers specifies a gluing of the total spaces, i.e., a gluing of $E_L \times S^2, E_M \times S^2, E_R \times S^2$ to E_T . Denote the result E_g . This space is a bundle over \mathcal{L}_3 with fiber denoted F_g .

FIGURE 6. The diagrams corresponding to the three codimension-one faces along which we glue.



Lemma 4.5. *The three principal faces $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ of E_T are pairwise disjoint.*

Proof. Recall that a face in the Axelrod–Singer/Fulton–MacPherson compactifications $C_n[M]$ is indexed by a set $\{S_1, \dots, S_k\}$ of subsets of $\{1, \dots, n\}$ which are nested or disjoint. The intersection of two faces indexed by $\{S_1, \dots, S_k\}$ and $\{S'_1, \dots, S'_\ell\}$ is the face indexed by $\{S_1, \dots, S_k, S'_1, \dots, S'_\ell\}$, if these sets are nested or disjoint. But in our case, the three faces are

indexed by $\{\{1, 4\}\}, \{\{2, 4\}\}, \{\{3, 4\}\}$, and each pair of sets is neither nested nor disjoint. Hence no pair of $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ intersect. \square

Now as indicated at the end of sections 4.1 and 4.2, the proof of Lemma 4.2 makes the main result of this section clear:

Proposition 4.6. *The glued total space E_g (or its fiber F_g) is a manifold with faces, hence a $\langle N \rangle$ -manifold for some N .* \square

Note that this implies of course that E_g is a manifold with corners.

5. FINISHING THE PROOF OF THEOREM 1

We are now ready to prove Theorem 1. Recall that we have maps from the spaces E_D to S^2 . Write these maps as

$$(16) \quad E_T \xrightarrow{\varphi_{14} \times \varphi_{24} \times \varphi_{34}} S^2 \times S^2 \times S^2$$

$$E_L \times S^2 \xrightarrow{\text{proj}_{S^2} \times \varphi_{14} \times \varphi_{23}} S^2 \times S^2 \times S^2$$

$$E_M \times S^2 \xrightarrow{\varphi_{12} \times \text{proj}_{S^2} \times \varphi_{34}} S^2 \times S^2 \times S^2$$

$$E_R \times S^2 \xrightarrow{\varphi_{14} \times \varphi_{23} \times \text{proj}_{S^2}} S^2 \times S^2 \times S^2$$

Recall that we defined the 2-forms θ_{ij} using $\omega_1, \omega_2, \omega_3 \in \Omega^2(S^2)$ with each ω_i cohomologous to a unit volume form. In the proofs of Lemma 3.5 and Lemma 3.8, we fixed each ω_i to have support in a small neighborhood in a different quadrant of the sphere. This ensured that the restriction of the pullback of $\omega_1 \times \omega_2 \times \omega_3$ to an infinite face (of any of the above four spaces) is zero.

Now note that the forms

$$\omega_1(-\theta_{13})(-\theta_{24}) = \omega_1\theta_{13}\theta_{24} \in \Omega^6(E_L \times S^2) \quad \text{and} \quad \theta_{14}\theta_{24}\theta_{34} \in \Omega^6(E_T)$$

agree on the common face

$$E_L \times S^2 \supset \mathfrak{S}_L \times S^2 \cong \mathfrak{S}_1 \subset E_T.$$

The minus signs above come from orientations of edges in the diagrams, which correspond to the order of i and j in θ_{ij} . Similarly, the forms

$$\theta_{12}\omega_2(-\theta_{34}) = -\theta_{12}\omega_2\theta_{34} \in \Omega^6(E_M \times S^2) \quad \text{and} \quad \theta_{14}\theta_{24}\theta_{34} \in \Omega^6(E_T)$$

agree on the common face

$$E_M \times S^2 \supset \mathfrak{S}_M \times S^2 \cong \mathfrak{S}_2 \subset E_T$$

and

$$\theta_{13}\theta_{24}\omega_3 \in \Omega^6(E_M \times S^2) \quad \text{and} \quad \theta_{14}\theta_{24}\theta_{34} \in \Omega^6 E_T$$

agree on the common face

$$E_R \times S^2 \supset \mathfrak{S}_R \times S^2 \cong \mathfrak{S}_3 \subset E_T$$

Let β be the 6-form on E_g obtained by gluing together of these four 6-forms.

Proposition 5.1. $\mu_{123} = \int_{F_g} \beta$.

Proof. By Corollary 3.2 and Lemma 4.4, we have

$$\begin{aligned} \mu_{123} &= \int_{F[2,1,1;0]} \theta_{13}\theta_{24} - \int_{F[1,2,1;0]} \theta_{12}\theta_{34} + \int_{F[1,1,2;0]} \theta_{13}\theta_{24} - \int_{F[1,1,1;1]} \theta_{14}\theta_{24}\theta_{34} \\ &= \int_{F[2,1,1;0] \times S^2} \omega_1 \theta_{13}\theta_{24} - \int_{F[1,2,1;0] \times S^2} \theta_{12}\omega_2\theta_{34} + \int_{F[1,1,2;0] \times S^2} \theta_{13}\theta_{24}\omega_3 - \int_{F[1,1,1;1]} \theta_{14}\theta_{24}\theta_{34} \\ &= \int_{F[2,1,1;0] \times S^2} \omega_1 \theta_{13}\theta_{24} + \int_{F[1,2,1;0] \times S^2} -\theta_{12}\omega_2\theta_{34} + \int_{F[1,1,2;0] \times S^2} \theta_{13}\theta_{24}\omega_3 - \int_{F[1,1,1;1]} \theta_{14}\theta_{24}\theta_{34} \\ &= \int_{F[2,1,1;0] \times S^2} \omega_1 \theta_{13}\theta_{24} + \int_{F[1,2,1;0] \times S^2} -\theta_{12}\omega_2\theta_{34} + \int_{F[1,1,2;0] \times S^2} \theta_{13}\theta_{24}\omega_3 + \int_{\overline{F[1,1,1;1]}} \theta_{14}\theta_{24}\theta_{34} \\ &= \int_{F_g} \beta \end{aligned}$$

where $\overline{F[1,1,1;1]}$ is $F[1,1,1;1] = F_T$ with its orientation reversed. \square

The vanishing of $\Phi_D^*(\omega_1 \times \omega_2 \times \omega_3)$ on infinite faces means that β represents a class in $H^6(E_g, \partial E_g)$. If we consider the integration over just one fixed link in the base \mathcal{L}_3 , then we can think of β as a top-dimensional class in $H^6(F_g, \partial F_g)$. We have seen that F_g is a manifold with corners, and ∂F_g is precisely its boundary, so there exists a fundamental class $[F_g, \partial F_g]$. In such a situation, integration is the same as pairing the cohomology class $[\beta]$ with the fundamental class $[F_g, \partial F_g]$:

$$\mu_{123} = \int_{F_g} \beta = \langle [\beta], [F_g, \partial F_g] \rangle$$

Since the ω_i represent integral cohomology classes, so does β , so the right-hand side can be viewed in integral (co)homology.

The maps (16) descend to a map $E_g \rightarrow S^2 \times S^2 \times S^2$ and, looking over just one link, to a map

$$(17) \quad F_g \longrightarrow S^2 \times S^2 \times S^2$$

with

$$H^6(F_g; \mathbb{Z}) \ni [\beta] \longleftarrow [\omega_1 \times \omega_2 \times \omega_3] \in H^6(S^2 \times S^2 \times S^2; \mathbb{Z})$$

where $[\beta]$ is really the image of $[\beta]$ under $H^6(F_g, \partial F_g) \rightarrow H^6(F_g)$, and where $[\omega_1 \times \omega_2 \times \omega_3]$ is a generator, since the ω_i represent generators of $H^2(S^2; \mathbb{Z})$.

Let $\mathcal{D} \subset S^2 \times S^2 \times S^2$ be the image of ∂F_g , thought of as the “degenerate locus”. (We know that \mathcal{D} is contained

$$((S^2)^3 \setminus (Q_1 \times Q_2 \times Q_3)) \cup \bigcup_{C=C_{xy}, C_{yz}, C_{xz}} ((C \times S^2 \times S^2) \cup (S^2 \times C \times S^2) \cup (S^2 \times S^2 \times C))$$

where C_{xy}, C_{yz}, C_{xz} are the three great circles in the coordinate planes.)

We can then rewrite the map (17) above as

$$F_g / \partial F_g \longrightarrow S^2 \times S^2 \times S^2 / \mathcal{D}.$$

Recall that $F_g = F_T \cup F_L \times S^2 \cup F_M \times S^2 \cup F_R \times S^2$. Recall also that the compactifications are homotopy-equivalent to their interiors and have collar neighborhoods of their boundaries. We now ignore the corner structure since we are purely in the realm of topology. Up to homeomorphism $F_T \cong I^3 \times (D^3 \setminus \{3 \text{ } D^3\text{'s}\})$; the three missing D^3 's correspond to the principal faces $F(\mathfrak{S}_1), F(\mathfrak{S}_2), F(\mathfrak{S}_3)$, which are homeomorphic to $I^3 \times S^2$. The other three $F_D \times S^2$ are homeomorphic to $I^4 \times S^2 = I^3 \times (I \times S^2)$, and the principal faces $F(\mathfrak{S}_D)$ correspond to the $I^3 \times S^2$ pieces of the boundary obtained by choosing one endpoint in the I factor. Thus $F_g \cong I^3 \times (D^3 \setminus \{3 \text{ } D^3\text{'s}\})$, since gluing F_L, F_M, F_R to F_T corresponds to filling in a neighborhood of three boundary components in the $(D^3 \setminus \{3 \text{ } D^3\text{'s}\})$ factor.

We extend the above map to the left, where CX denotes the cone on X :

$$D^6 \cong I^3 \times ((D^3 \setminus \{3 \text{ } D^3\text{'s}\}) \cup \bigcup_1^3 CS^2) \longrightarrow D^6 / \partial D^6 \longrightarrow F_g / \partial F_g \longrightarrow S^2 \times S^2 \times S^2 / \mathcal{D}$$

The first map is just the quotient by the boundary. The middle map is the quotient by the image of $I^3 \times \{3 \text{ cone points}\}$ under the first map. This image in $D^6 / \partial D^6$ is $\bigvee_1^3 S^3$, so this middle map is the quotient

$$\bigvee_1^3 S^3 \hookrightarrow S^6 \twoheadrightarrow F_g / \partial F_g$$

which induces isomorphisms in H^6 and H_6 . Under the isomorphism in H^6 , $[F_g, \partial F_g]$ corresponds to a fundamental class $[S^6]$. Thus the triple linking number $\mu_{123} = \langle [\beta], [F_g, \partial F_g] \rangle$ is given by the pairing of the fundamental class of S^6 with a cohomology class which is pulled back via the left-hand map below and which maps to a generator of $H^6(S^2 \times S^2 \times S^2; \mathbb{Z})$ via

the right-hand map:

$$S^6 \longrightarrow S^2 \times S^2 \times S^2/\mathcal{D} \longleftarrow S^2 \times S^2 \times S^2$$

This proves Theorem 1. \square

The constraints we have imposed on the ω_i imply that $\alpha = \omega_1 \times \omega_2 \times \omega_3$ has support in a connected component of $S^2 \times S^2 \times S^2 \setminus \mathcal{D}$ (or even a subset of $S^2 \times S^2 \times S^2$ diffeomorphic to a 6-ball). Quotienting by the complement of the support of α thus gives a space $S(\alpha)$ which has a fundamental class. It is not hard to see that the degree of the composition

$$S^6 \longrightarrow S^2 \times S^2 \times S^2/\mathcal{D} \longrightarrow S(\alpha)$$

is the triple linking number μ_{123} . So we have exhibited it as a degree, as promised.

6. INTEGRATION ALONG THE FIBER VIA PONTRJAGIN–THOM

This section reviews a classical result relating integration along the fiber to a Pontrjagin–Thom construction. We proceed with a general situation, avoiding any possible simplifications in our particular example of the triple linking number. This is because we expect these results to be quite useful in our future work. We begin with the case where the fiber has no boundary, and then treat the case of a fiber with boundary. The latter case proceeds very similarly to the former, but we start with the boundaryless case for ease of readability.

6.1. Warmup: the case of boundaryless fibers. Suppose $F \rightarrow E \rightarrow B$ is a fiber bundle of compact manifolds (without boundary). Then integration along the fiber is a chain map (cf. equation (14) as well as [6], p. 62) and hence induces a map in cohomology

$$\int_F : H^p(E) \longrightarrow H^{p-k}(B).$$

Since we will use de Rham cohomology (or compare other constructions to those in de Rham cohomology) all (co)homology will be taken with real coefficients for the rest of this section.

The above map can also be produced as follows. Since E is a compact manifold, we can embed it in some \mathbb{R}^N and use this to embed the bundle into a trivial bundle:

$$\begin{array}{ccc} E & \xhookrightarrow{e} & \mathbb{R}^N \times B \\ & \searrow & \swarrow \\ & B & \end{array}$$

Then E has a tubular neighborhood $\eta(E)$ in $\mathbb{R}^N \times B$ which is diffeomorphic to the normal bundle $\nu = \nu(e)$ of the embedding e . Taking the quotient by the complement gives

$$\begin{array}{ccc} \mathbb{R}^N \times B & \xrightarrow{\quad} & \mathbb{R}^N \times B / (\mathbb{R}^N \times B - \eta(E)) \cong E^\nu \\ & \searrow & \nearrow \tau \\ & \Sigma^N B_+ & \end{array}$$

Here E^ν denotes the Thom space $D(\nu)/S(\nu)$ of the normal bundle $\nu \rightarrow E$ and $\Sigma^N B_+$ is the N -fold suspension of $B_+ = B \sqcup \{*\}$. The map above factors through $\Sigma^N B_+$ because that space is just the one-point compactification of $B \times \mathbb{R}^N$, and because E is compact. We consider the map in cohomology induced by $\tau : \Sigma^N B_+ \rightarrow E^\nu$.

Lemma 6.1. *Given a bundle $F \longrightarrow E \xrightarrow{\pi} B$ with all three spaces compact manifolds without boundary and $k = \dim F$, then $\int_F : H^p(E) \rightarrow H^{p-k}(B)$ agrees with the composition*

$$H^{p+k}(E) \xrightarrow{\text{Thom} \cong} \tilde{H}^{p+N}(E^\nu) \xrightarrow{\tau^*} \tilde{H}^{p+N}(\Sigma^N B_+) \xrightarrow{\Sigma \cong} H^p(B)$$

induced by the Thom collapse map, the Thom isomorphism, and the suspension isomorphism.

Proof. We first check that integration along the fiber is Poincaré dual to the map π^* induced by $\pi : E \rightarrow B$. That is, we check that $\int_F \beta \cap [B] = \pi_*(\beta \cap [E])$ for any class $\beta \in H^*E$.

Let $\langle \cdot, \cdot \rangle$ denote the pairing between cohomology and homology. Let $\alpha \in H^{m-p}B$ where $m = \dim B$. Using the relationship between cap and cup products and the “naturality” property of cap products, we have

$$\begin{aligned} \langle \alpha, \int_F \beta \cap [B] \rangle &= \langle \alpha \cup (\int_F \beta), [B] \rangle \\ &= \int_B \alpha \wedge (\int_F \beta) \\ &= \int_E \pi^* \alpha \wedge \beta \\ &= \langle \pi^* \alpha \cup \beta, [E] \rangle \\ &= \langle \pi^* \alpha, \beta \cap [E] \rangle \\ &= \langle \alpha, \pi_*(\beta \cap [E]) \rangle \end{aligned}$$

The equality of the integrals in the second and third lines follows from a Fubini Theorem. (See [9], p. 307-309; see also [1] for a characterization of fiber integration in terms of Poincaré duality.) Since this holds for any $\alpha \in H^{m-p}B$, we must have $\int_F \beta \cap [B] = \pi_*(\beta \cap [E])$. Thus

the following diagram commutes:

$$(18) \quad \begin{array}{ccc} H^{p+k}(E) & \xrightarrow{-\cap[E]} & H_{m-p}(E) \\ \int_F \downarrow & & \downarrow \pi_* \\ H^p(B) & \xleftarrow{(-\cap[B])^{-1}} & H_{m-p}(B) \end{array}$$

Next we claim the diagram below commutes:

$$\begin{array}{ccc} H^{p+k}(E) & \xrightarrow[\cong]{-\cap[E]} & H_{m-p}(E) \\ \text{Thom} \cong \downarrow & & \downarrow \cong \\ H^{p+N}(D(\nu), S(\nu)) & \xrightarrow[\cong]{-\cap[D(\nu), S(\nu)]} & H_{m-p}(D(\nu)) \\ i^* \uparrow \cong & & \downarrow i_* \\ H^{p+N}(\mathbb{R}^N \times B, \mathbb{R}^N \times B - \eta(E)) & \xrightarrow{-\cap i_*[D(\nu), S(\nu)]} & H_{m-p}(\mathbb{R}^N \times B) \\ \tau^* \downarrow & & \uparrow \cong \\ H^{p+N}(D^N \times B, \partial D^N \times B) & \xrightarrow[\cong]{-\cap[D^N \times B, \partial D^N \times B]} & H_{m-p}(D^N \times B) \\ \Sigma \cong \downarrow & & \uparrow \cong \\ H^p(B) & \xrightarrow[\cong]{-\cap[B]} & H_{m-p}(B) \end{array}$$

Commutativity of the first square at the top is just the fact that the Thom class of a disk bundle is the Poincaré dual to the zero section. (In fact, one can take this as the definition of the Thom class.) The second square commutes by the “naturality” of the cap product, applied to $i : (D(\nu), S(\nu)) \rightarrow (\mathbb{R}^N \times B, \mathbb{R}^N \times B - \eta(E))$; i^* is an isomorphism by excision. In the third square, D^N is a ball in \mathbb{R}^N large enough to contain the image of the embedding e . The commutativity of this square comes from the “naturality” of the cap product applied to the collapse map $\tau : (D^N \times B, \partial D^N \times B) \rightarrow (\mathbb{R}^N \times B, \mathbb{R}^N \times B - \eta(E))$. The bottom square commutes up to a sign by the relationship between cap and cross products. By choosing the suspension isomorphism appropriately, this diagram will commute on the nose.

Finally notice that $(-\cap[B])^{-1} \circ \pi_* \circ (-\cap[E])$ is the same as going clockwise around the above diagram, from the upper-left corner to the lower-left corner. The composition induced by the Thom collapse map corresponds to going down the left-hand side of the diagram. \square

6.2. Fibers with boundary. Now let $F \rightarrow E \rightarrow B$ be a fiber bundle with F a k -dimensional compact manifold with boundary, or even a manifold with faces. Continue to assume that B is a compact manifold (without boundary). Considering the boundary of the fiber $\partial F \subset F$ gives us a bundle $\partial E \rightarrow B$ which is a sub-bundle of $E \rightarrow B$. Let β be

a $(p+k)$ -form on E . Suppose the restriction of β to ∂E is zero. Then the corresponding cohomology class $\beta \in H^{p+k}(E)$ comes from a class in $H^{p+k}(E, \partial E)$.

Alternatively, E has a collar neighborhood $[0, 1) \times \partial E$ of its boundary, so E is homotopy-equivalent to the complement of a collar, say $[0, \frac{1}{2}) \times \partial E$. Under the homotopy, the form β becomes a form on the interior of E with compact support. Thus we may view β as an element in cohomology with compact support, $\beta \in H_c^{p+k} E$.

The vanishing on ∂E of β implies by Stokes' Theorem that integration along the fiber produces a well defined cohomology class $\int_F \beta \in H^{p-k}(B)$ (again cf. equation (14) and [6], p. 62).

Since F is a manifold with faces it can be embedded neatly in some Euclidean space with corners $\mathbb{R}^{\langle N \rangle, M} := [0, \infty)^N \times \mathbb{R}^M$. Recall that a neat embedding has a well defined normal bundle.

Since all the corner structure in E comes from that in F , we can neatly embed the bundle $E \rightarrow B$ into a trivial bundle

$$\begin{array}{ccc} E & \xrightarrow{e} & \mathbb{R}^{\langle N \rangle, M} \times B \\ & \searrow & \swarrow \\ & B & \end{array}$$

Once again a tubular neighborhood $\eta(E) \subset \mathbb{R}^{\langle N \rangle, M} \times B$ is diffeomorphic to the normal bundle $\nu = \nu(e)$ of the embedding e . Taking the quotient by the complement gives a Pontrjagin–Thom collapse map

$$\begin{array}{ccc} \mathbb{R}^{\langle N \rangle, M} \times B & \xrightarrow{\quad} & \mathbb{R}^{\langle N \rangle, M} \times B / (\mathbb{R}^{\langle N \rangle, M} \times B - \eta(E)) \cong E^\nu \\ & \searrow & \nearrow \tau \\ & C^N \Sigma^M B_+ & \end{array}$$

The map above factors through $C^N \Sigma^M B_+$ (where C^N denotes N -fold cone) because that is just the one-point compactification of $\mathbb{R}^{\langle N \rangle, M} \times B$. Restrict the embedding e to $\partial E \subset E$, and let $\partial E^\nu \subset E^\nu$ be the result of the corresponding restriction of the collapse map. If we let

$$\partial(C^N \Sigma^M B_+) = \partial([0, \infty)^N) \times \mathbb{R}^M \times B \subset [0, \infty)^N \times \mathbb{R}^M \times B \subset C^N \Sigma^M B_+$$

then clearly

$$e : \partial E \hookrightarrow \partial(C^N \Sigma^M B_+)$$

so restricting the collapse map above gives a map of pairs

$$(C^N \Sigma^M B_+, \partial(C^N \Sigma^M B_+)) \rightarrow (E^\nu, \partial E^\nu).$$

The resulting map on quotients is $\tau : \Sigma^{N+M} B_+ \rightarrow E^\nu / \partial E^\nu$.

Lemma 6.2. *Suppose as above that $F \rightarrow E \rightarrow B$ is a fiber bundle with B a compact manifold, F a k -dimensional compact manifold with boundary. Let $\beta \in \Omega^{p+k}(E)$ be a form whose restriction to ∂E is zero. Then the class $\int_F \beta$ is precisely the image of β under the composition*

$$H^{p+k}(E, \partial E) \xrightarrow{\text{Thom} \cong} \tilde{H}^{p+N+M}(E^\nu / \partial E^\nu) \xrightarrow{\tau^*} \tilde{H}^{p+N+M}(\Sigma^{N+M} B_+) \xrightarrow{\Sigma \cong} H^p(B)$$

induced by the Thom collapse map together with the Thom isomorphism and suspension isomorphism.

Proof. The proof is very similar to the proof of Lemma 6.1. We have $\beta \in H^{p+k}(E, \partial E) \cong H_c^{p+k}(E - \partial E)$ and $\int_F \beta \in H_c^p(B) \cong H^p(B)$ since B is compact.

As before let $\alpha \in H^{m-p} B$ where $m = \dim B$. We have

$$\begin{aligned} \langle \alpha, \int_F \beta \cap [B] \rangle &= \langle \alpha \cup (\int_F \beta), [B] \rangle \\ &= \int_B \alpha \wedge (\int_F \beta) \\ &= \int_E \pi^* \alpha \wedge \beta \end{aligned}$$

The last line holds because we are considering the pairings $\int_B : H^{m-p} B \otimes H_c^p B \rightarrow \mathbb{R}$ and $\int_E : H^{m-p}(E - \partial E) \otimes H_c^{p+k}(E - \partial E) \rightarrow \mathbb{R}$. Continuing we have

$$\begin{aligned} \langle \alpha, \int_F \beta \cap [B] \rangle &= \int_E \pi^* \alpha \wedge \beta \\ &= \langle \pi^* \alpha \cup \beta, [E, \partial E] \rangle \\ &= \langle \pi^* \alpha, \beta \cap [E, \partial E] \rangle \\ &= \langle \alpha, \pi_*(\beta \cap [E, \partial E]) \rangle \end{aligned}$$

Since this holds for any $\alpha \in H^{m-p} B$, we must have $\int_F \beta \cap [B] = \pi_*(\beta \cap [E])$. Thus the following diagram commutes:

$$(19) \quad \begin{array}{ccc} H^{p+k}(E) & \xrightarrow{-\cap[E, \partial E]} & H_{m-p}(E) \\ \int_F \downarrow & & \pi_* \downarrow \\ H^p(B) & \xleftarrow{(-\cap[B])^{-1}} & H_{m-p}(B) \end{array}$$

Next we claim the diagram below commutes:

$$\begin{array}{ccc}
H^{p+k}(E, \partial E) & \xrightarrow[\cong]{-\cap[E, \partial E]} & H_{m-p}(E) \\
\text{relative Thom } \cong \downarrow & & \cong \downarrow \\
H^{p+N+M}(D(\nu), S(\nu) \cup D(\nu|_{\partial E})) & \xrightarrow[\cong]{-\cap[D(\nu), S(\nu) \cup D(\nu|_{\partial E})]} & H_{m-p}(D(\nu)) \\
i^* \uparrow \cong & & \downarrow i_* \\
H^{p+N+M}(\mathbb{R}^{\langle N \rangle, M} \times B, \mathbb{R}^{\langle N \rangle, M} \times B - (\eta(E) \cup \partial E)) & \xrightarrow[-\cap i_*[D(\nu), S(\nu) \cup D(\nu|_{\partial E})]]{} & H_{m-p}(\mathbb{R}^{\langle N \rangle, M} \times B) \\
\tau^* \downarrow & & \uparrow \cong \\
H^{p+N+M}(D^{\langle N \rangle, M} \times B, \partial D^{\langle N \rangle, M} \times B) & \xrightarrow[\cong]{-\cap[D^{\langle N \rangle, M} \times B, \partial D^{\langle N \rangle, M} \times B]} & H_{m-p}(D^{\langle N \rangle, M} \times B) \\
\Sigma \cong \downarrow & & \uparrow \cong \\
H^p(B) & \xrightarrow[\cong]{-\cap[B]} & H_{m-p}(B)
\end{array}$$

Commutativity of the first square at the top is the expression of the *relative* Thom class as Poincaré dual to $[E, \partial E]$ (the relative zero section). The second and third squares commute by “naturality” of the cap product as before. With the appropriate choice of sign in the suspension isomorphism, the bottom square will commute by the relationship between cap and cross products. Now considering this large commutative rectangle and the previous commutative square, we have proven the lemma. \square

7. THE TRIPLE LINKING NUMBER VIA PONTRJAGIN–THOM

In this section, we finish the proof of Theorem 2. In order to do this, we first need to embed our glued bundle $E_g \rightarrow \mathcal{L}_3$ into a trivial bundle in a way that preserves the corner structure.

7.1. Neatly embedding the glued total space.

Lemma 7.1. *The manifold with faces E_g can be embedded into a trivial bundle over \mathcal{L}_3 in such a way that it has a well defined tubular neighborhood and normal bundle.*

Proof. Each of the four spaces E_D (for the four diagrams $D = L, M, R, T$) can be neatly embedded

$$E_D \hookrightarrow C_4[\mathbb{R}^3] \times \mathcal{L}_3$$

using its definition as the pullback in square (13) and the fact that all the corner structure in E_D comes from that in $C_4[\mathbb{R}^3]$. Lemma 3.2.1 of [13] shows that the spaces $C_n[\mathbb{R}^d]$ can be

given the structure of $\langle N \rangle$ -manifolds. So we have neat embeddings

$$(20) \quad \begin{array}{ccccc} E_T & \hookrightarrow & C_4[\mathbb{R}^3] \times \mathcal{L}_3 & \hookrightarrow & \mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3 \\ & & & \searrow & \downarrow \\ & & & & \mathcal{L}_3 \end{array}$$

$$(21) \quad \begin{array}{ccccc} E_L & \hookrightarrow & C_4[\mathbb{R}^3] \times \mathcal{L}_3 & \hookrightarrow & \mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3 \\ & & & \searrow & \downarrow \\ & & & & \mathcal{L}_3 \end{array}$$

We reverse the orientation on the top copy of $\mathbb{R}^M \times [0, \infty)^N$ and then glue the top copy of $\mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3$ to the bottom one in such a way that the principal face of the top $C_4[\mathbb{R}^3]$ containing \mathfrak{S}_1 is glued via the identity map to the principal face of the bottom $C_4[\mathbb{R}^3]$ containing \mathfrak{S}_L . (So far E_T has not been glued to E_L in any “nice” way.)

Let $\eta(\mathfrak{S}_L) \cong \mathfrak{S}_L \times [0, 1)$ denote a collar neighborhood of \mathfrak{S}_L in E_L . We have the identification $g : \mathfrak{S}_L \times S^2 \xrightarrow{\cong} \mathfrak{S}_1$, and we can further identify $\eta(\mathfrak{S}_L) \times S^2$ with a collar $\eta(\mathfrak{S}_1)$ of \mathfrak{S}_1 in E_T . Consider the restriction of (20) to $\eta(\mathfrak{S}_1)$ and the fact that these neat embeddings are perpendicular at the boundaries of $\mathbb{R}^M \times [0, \infty)^N$; this allows us to smoothly extend the bottom embedding to $E_L \cup (\eta(\mathfrak{S}_L) \times S^2)$ by “doubling” it near the boundary:

$$(22) \quad \begin{array}{ccccc} E_T \cup_g (E_L \cup (\eta(\mathfrak{S}_L) \times S^2)) & \hookrightarrow & (C_4[\mathbb{R}^3] \cup C_4[\mathbb{R}^3]) \times \mathcal{L}_3 & \hookrightarrow & (\mathbb{R}^M \times [0, \infty)^N \cup \mathbb{R}^M \times [0, \infty)^N) \times \mathcal{L}_3 \\ & & & \searrow & \downarrow \\ & & & & \mathcal{L}_3 \end{array}$$

Since the interior of the fiber F_L is an open subset of Euclidean space, F_L has a trivial normal bundle. The normal bundle to $E_L \hookrightarrow \mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3$ is the normal bundle along the fiber. Thus a tubular neighborhood of E_L in $\mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3$ can be identified with $E_L \times \mathbb{R}^p$ for some p . Considering the dimensions of F_L and $C_4[\mathbb{R}^3]$, we see that $p \geq 8$.

Consider the restriction

$$\eta(\mathfrak{S}_L) \times S^2 \hookrightarrow \mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3$$

of the embedding (22) to a collar of the face along which we glued. We may assume that its image lies in a tubular neighborhood of E_L : in fact, we can shrink the embedding of the S^2 factor in $\eta(\mathfrak{S}_1) \cup_g (\eta(\mathfrak{S}_L) \times S^2) \cong \mathfrak{S}_L \times S^2 \times (-1, 1)$ along the interval $(-1, 0]$ to achieve this. Thus over each link in \mathcal{L}_3 we have an embedding

$$F(\mathfrak{S}_L) \times S^2 \hookrightarrow F(\mathfrak{S}_L) \times \mathbb{R}^p.$$

Let $F(\mathfrak{S}_L)$ denote the fiber of the face \mathfrak{S}_L over a link in \mathcal{L}_3 . By making p sufficiently large (by making M above sufficiently large), we can find a smooth isotopy from the embedding above to an embedding

$$F(\mathfrak{S}_L) \times S^2 \hookrightarrow F(\mathfrak{S}_L) \times \mathbb{R}^p$$

induced by a standard embedding $S^2 \hookrightarrow \mathbb{R}^p$. If we take the trace of this isotopy along part of the collar (say, along $[\frac{1}{3}, \frac{2}{3}]$), we can extend outside $F(\mathfrak{S}_L) \times S^2 \times [0, \frac{2}{3}]$ using the standard embedding. This gives an embedding $F_L \times S^2 \hookrightarrow F_L \times \mathbb{R}^p$ which on the face $F(\mathfrak{S}_L)$ agrees with the one gotten by restricting (22). This gives an embedding of the whole space

$$E_L \times S^2 \hookrightarrow E_L \times \mathbb{R}^p \hookrightarrow \mathbb{R}^M \times [0, \infty)^N \times \mathcal{L}_3$$

which on $E_L \cup (\eta(\mathfrak{S}_L) \times S^2)$ agrees with (22). Thus we have embedded

$$E_T \cup_g (E_L \times S^2) \hookrightarrow ((\mathbb{R}^M \times [0, \infty)^N) \cup (\mathbb{R}^M \times [0, \infty)^N)) \times \mathcal{L}_3.$$

Our construction of the embedding in (22) gives a smooth embedding of the collar of the face we glued, where the manifold-with-faces structure on the glued space is the one indicated in the proof of Lemma 4.2. Furthermore, the embedding of each piece is neat and hence has a tubular neighborhood which can be identified with its normal bundle. So the embedding above has a tubular neighborhood diffeomorphic to its normal bundle.

Continuing in a similar manner, we embed

$$(23) \quad E_g \hookrightarrow (\bigcup_1^4 \mathbb{R}^M \times [0, \infty)^N) \times \mathcal{L}_3$$

where each of the last 3 copies of $\mathbb{R}^M \times [0, \infty)^N$ is glued to the first copy along a different codimension-1 face of the first copy. The result is a subspace of $\mathbb{R}^M \times \mathbb{R}^N$. We saw that the domain is a manifold with faces, but the codomain is not (necessarily) even a manifold with corners, since the three pairs of faces we glue along are not disjoint there. Nonetheless, as in Lemma 4.5, the faces of the E_D which we glue are disjoint. Thus near any point in any E_D , the embedding looks the same as in the case where we glue just one pair of spaces. Thus, as in that case, the embedding has a well defined tubular neighborhood, diffeomorphic to its normal bundle. \square

7.2. Finishing the proof of Theorem 2. Proposition 5.1 described the triple linking number μ_{123} as $\int_{F_g} \beta$, the integral over the fiber of a glued configuration space bundle of a certain 6-form β . We now compare this to the Pontrjagin–Thom construction.

Let $b_{M,N}(R)$ denote the intersection of a radius R ball around the origin in \mathbb{R}^{M+N} with $\mathbb{R}^M \times [0, \infty)^N$. Let $B_{M,N}(R)$ denote the result of gluing four copies of this sector $b_{M,N}(R)$ in the same way that the four copies of $\mathbb{R}^M \times [0, \infty)^N$ were glued. The embedding (23) gives

rise to a Pontrjagin–Thom collapse map, which induces a map of pairs

$$(B_{M,N}(R) \times \mathcal{L}_3, *) \longrightarrow (B_{M,N}(R) \times \mathcal{L}_3, B_{M,N}(R) \times \mathcal{L}_3 - \eta(E_g))$$

for sufficiently large R , where $\eta(E_g)$ is a tubular neighborhood of E_g and where the basepoint $*$ is a point of distance R from the origin in $B_{M,N}(R)$. The collapse map further descends to the map of pairs below

$$\tau : (B_{M,N}(R) \times \mathcal{L}_3, \partial(B_{M,N}(R)) \times \mathcal{L}_3) \longrightarrow (B_{M,N}(R) \times \mathcal{L}_3, (B_{M,N}(R) \times \mathcal{L}_3 - \eta(E_g)) \cup \partial E_g)$$

which on the level of quotients is

$$\tau : \Sigma^{M+N} \mathcal{L}_3 \longrightarrow E_g^{\nu_{M,N}} / \partial E_g^{\nu_{M,N}}$$

where $E_g^{\nu_{M,N}}$ is the Thom space of the normal bundle $\nu_{M,N} \rightarrow E_g$ of our embedding of E_g in a trivial bundle.

We have already seen that, by our choices of spherical forms and vanishing arguments for infinite faces, the form β represents a class $[\beta] \in H^6(E_g, \partial E_g)$. We claim that by Lemma 6.2, the image $\tau^*[\beta]$ of $[\beta]$ under the map in cohomology induced by the collapse map above is precisely $\int_{F_g} \beta = \mu_{123}$. Strictly, Lemma 6.2 only shows this when the base is a finite-dimensional compact manifold. However, we can consider finite-dimensional compact submanifolds of \mathcal{L}_3 . Since the $\tau^*\beta$ and $\int_{F_g} \beta$ agree over any such submanifold, they are indeed equal. This proves Theorem 2. \square

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DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI, USA

E-mail address: robink@math.brown.edu